Fibred sites and existential toposes

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Aim of the talk

The aim of the talk is to present a way for representing relative toposes which naturally generalizes the construction of the topos of sheaves on a locale, and which is particularly effective for describing in a simple way the morphisms between relative toposes.

Recall that, given locales L and L', the morphisms $\mathbf{Sh}(L) \to \mathbf{Sh}(L')$ correspond exactly to the locale homomorphisms $L \to L'$.

Our representation will be based on the concept of existential fibred site.

By using this notion, we shall be able to describe the morphisms between two relative toposes as morphisms between the associated existential fibred sites.

Relative sites

Given an indexed category $\mathbb{D} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ and a Grothendieck topology J on $\mathcal{C}$, we shall denote by

$$p_{\mathbb{D}} : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$$

the fibration associated with $\mathbb{D}$ through the Grothendieck construction.

Given a Grothendieck topology J on $\mathcal{C}$, the Giraud topology $J_{\mathbb{D}}$ on $\mathcal{G}(\mathbb{D})$ is the smallest topology which makes the projection functor $p_{\mathbb{D}} : \mathcal{G}(\mathbb{D}) \to \mathcal{C}$ a comorphism of sites to $(\mathcal{C}, J)$.

Definition

Let $(\mathcal{C}, J)$ be a small-generated site. A relative site over $(\mathcal{C}, J)$ is a site of the form $(\mathcal{G}(\mathbb{D}), J')$, where $\mathbb{D}$ is a $\mathcal{C}$-indexed category and J' is a Grothendieck topology on $\mathcal{G}(\mathbb{D})$ containing the Giraud topology $J_{\mathbb{D}}$.

Any relative site $(\mathcal{G}(\mathbb{D}), J')$ is endowed with the structure comorphism of sites $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), J') \to (\mathcal{C}, J)$.

Relative toposes

Definition

Let $(\mathcal{C}, J)$ be a small-generated site. A relative topos over $\mathbf{Sh}(\mathcal{C}, J)$ is a Grothendieck topos $\mathcal{E}$, together with a geometric morphism $p : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$.

Theorem

Let $(\mathcal{C}, J)$ be a small-generated site. Then any relative site over $(\mathcal{C}, J)$ yields a relative topos over $\mathbf{Sh}(\mathcal{C}, J)$; more precisely, any relative site

$$p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), J') \to (\mathcal{C}, J)$$

induces the relative topos

$$C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), J') \to \mathbf{Sh}(\mathcal{C}, J),$$

where $C_{p_{\mathbb{D}}}$ is the geometric morphism induced by $p_{\mathbb{D}}$, regarded as a comorphism of sites $(\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$.

Conversely, any relative topos $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ is of the form $C_{p_{\mathbb{D}}}$ for some relative site $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), J') \to (\mathcal{C}, J)$ (for instance, one can take $p_{\mathbb{D}}$ to be the canonical relative site of f, as defined below).

The canonical stack of a geometric morphism

Definition

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. The relative topology of f is the Grothendieck topology on the category $(1_{\mathcal{F}} \downarrow f^*)$ induced by the canonical topology on $\mathcal{F}$ via the projection functor $\pi_{\mathcal{F}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{F}$.

Theorem

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then the canonical projection functor

$$\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$$

is a comorphism of sites

$$((1_{\mathcal{F}} \downarrow f^*), J_f) \to (\mathcal{E}, J_{\mathcal{E}}^{\text{can}})$$

such that $f = C_{\pi_{\mathcal{E}}}$.

The functor $\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$ is actually a stack on $\mathcal{E}$, which we call the canonical stack of f: from an indexed point of view, this stack sends any object E of $\mathcal{E}$ to the topos $\mathcal{F}/f^*(E)$ and any arrow $u : E' \to E$ to the pullback functor $u^* : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$.

The comorphism of sites $\pi_{\mathcal{E}} : ((1_{\mathcal{F}} \downarrow f^*), J_f) \to (\mathcal{E}, J_{\mathcal{E}}^{\text{can}})$ is called the canonical relative site of f.

Relative Diaconescu's equivalence

Recall that, if $(\mathcal{C}, J)$ is a cartesian site, we have, for any topos $\mathcal{E}$, an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Cart}_J(\mathcal{C}, \mathcal{E}),$$

where $\mathbf{Cart}_J(\mathcal{C}, \mathcal{E})$ is the category of cartesian functors $\mathcal{C} \to \mathcal{E}$ which are J-continuous.

The following result is a relative generalization of this:

Theorem

Let $(\mathcal{C}, J)$ be a small-generated site, where $\mathcal{C}$ is a cartesian category, $\mathbb{D} : \mathcal{C}^{\text{op}} \to \text{Cart}$ a pseudofunctor, K a Grothendieck topology on $\mathcal{G}(\mathbb{D})$ containing the Giraud topology $J_{\mathbb{D}}$, $A : \mathcal{C} \to \mathcal{F}$ a cartesian J-continuous functor inducing a geometric morphism $f : \mathcal{F} \to \mathbf{Sh}(\mathcal{C}, J)$. Then we have an equivalence of categories

$$\mathbf{Geom}_{\mathbf{Sh}(\mathcal{C},J)}([f], [C_{p_{\mathbb{D}}}]) \simeq \mathbf{Fib}^{\text{cart,cov}}_{\mathcal{C}}((p_{\mathbb{D}}, K), (1_{\mathcal{F}} \downarrow A), J_f|_{(1_{\mathcal{F}} \downarrow A)}),$$

where $\mathbf{Fib}^{\text{cart,cov}}_{\mathcal{C}}((p_{\mathbb{D}}, K), (1_{\mathcal{F}} \downarrow A), J_f|_{(1_{\mathcal{F}} \downarrow A)})$ is the category of morphisms of fibrations over $\mathcal{C}$ which are cartesian at each fiber and cover-preserving.

Two corollaries

Corollary

Let $f : \mathcal{F} \to \mathcal{E}$ and $f' : \mathcal{F}' \to \mathcal{E}$ be geometric morphisms towards the same base topos $\mathcal{E}$. Then we have an equivalence of categories

$$\mathbf{Geom}_{\mathcal{E}}([f],[f']) \simeq \mathbf{Fib}_{\mathcal{E}}^{\text{cart,cov}}(((1_{\mathcal{F}'} \downarrow f'^{*}), J_{f'}), ((1_{\mathcal{F}} \downarrow f^{*}), J_{f})),$$

where $\mathbf{Fib}_{\mathcal{E}}^{\text{cart,cov}}(((1_{\mathcal{F}'} \downarrow f'^{*}), J_{f'}), ((1_{\mathcal{F}} \downarrow f^{*}), J_{f}))$ is the category of morphisms of fibrations over $\mathcal{E}$ which are cartesian at each fiber and cover-preserving.

Corollary

Let $\mathcal{E}$ be a Grothendieck topos and L, L' internal locales in $\mathcal{E}$. Then we have an equivalence of categories

$$\mathbf{Geom}_{\mathcal{E}}(\mathbf{Sh}_{\mathcal{E}}(L), \mathbf{Sh}_{\mathcal{E}}(L')) \simeq \mathbf{Loc}_{\mathcal{E}}(L, L'),$$

where $\mathbf{Loc}_{\mathcal{E}}(L, L')$ is the category of morphisms of internal locales from L to L' in $\mathcal{E}$.

Fibred sites

Definition

Let $(\mathcal{C}, J)$ be a small-generated site.

(a) A fibred site over $\mathcal{C}$ is an indexed category $L : \mathcal{C}^{\text{op}} \to \text{Cat}$ taking values in the category of small-generated sites and morphisms of sites between them; we shall denote by J^L_e the Grothendieck topology on the fiber $L(e)$.

(b) A fibred site over $(\mathcal{C}, J)$ is a fibred site over $\mathcal{C}$ which is J-reflecting in the sense that for any J-covering family S on an object c of $\mathcal{C}$ and any family T of arrows with common codomain in the category $L(c)$, if $L(f)(T)$ is $J^L_{\text{dom}(f)}$-covering in the category $L(\text{dom}(f))$ for every $f \in S$ then T is J^L_c-covering.

(c) A fibred site $L : \mathcal{C}^{\text{op}} \to \text{Cat}$ over $(\mathcal{C}, J)$ is said to be existential if for any arrow $a : E' \to E$ in $\mathcal{C}$, the transition functor $L(a) : L(E) \to L(E')$ has a left adjoint, denoted $\exists_a : L(E') \to L(E)$ (which is therefore a comorphism of sites $(L(E'), J^L_{E'}) \to (L(E), J^L_E)$), and the following two conditions (where, for any f, η_f denotes the unit of the adjunction $\exists_f \dashv L(f)$) are satisfied:

Existential fibred sites

(i) Relative Beck-Chevalley condition:
For any arrows $c : V \to Z$ and $d : W \to Z$ in $\mathcal{C}$ with common codomain and any $l \in L(V)$, the family of arrows

$$\{\overline{\widetilde{L(a)(\eta_c(l))}} : (\exists_b)(L(a)(l)) \to L(d)(\exists_c(l)) \mid (a, b) \in B_{(c,d)}\}$$

is J^L_W-covering, where $B_{(c,d)}$ is the collection of spans $(a : U \to V, b : U \to W)$ such that $c \circ a = d \circ b$

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ W & \xrightarrow{d} & Z \end{array}$$

and $\overline{\widetilde{L(a)(\eta_c(l))}}$ is the transpose of the arrow

$$\widetilde{L(a)(\eta_c(l))} : L(a)(l) \to L(b)(L(d)(\exists_c(l)))$$

given by the composite of the arrow $L(a)(\eta_c(l))$ with the inverse of the isomorphism $L(b)(L(d)(\exists_c(l))) \to L(a)(L(c)(\exists_c(l)))$ resulting from the equality $c \circ a = d \circ b$ in light of the pseudofonctoriality of L.

Existential fibred sites

(ii) Relative Frobenius condition: For any arrows $f : E \to E'$ in $\mathcal{C}$, any $l \in L(E')$ and any arrow $\alpha : l' \to \exists_f(l)$, the family of arrows $\{\overline{\delta} : \exists_f(m) \to l' \mid (\delta, \rho) \in Q_{(f,\alpha)}\}$ is $J^L_{E'}$-covering, where $Q_{(f,\alpha)}$ is the collection of span of arrows $(\rho : m \to l, \delta : m \to L(f)(l'))$ in $L(E)$ which make the rectangle

$$
\begin{array}{ccc}
m & \xrightarrow{\quad\rho\quad} & l \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle \eta_f(l)} \\
L(f)(l') & \xrightarrow[L(f)(\alpha)]{} & L(f)(\exists_f(l))
\end{array}
$$

commute.

Remark

One can generalize the notion of fibred site by simply requiring the transition morphisms to be cover-preserving (rather than morphisms of sites). The theorem below about the existential topology (see below) remains valid, but the results below on fibers of existential toposes require the stronger assumptions.

The fibred site of a geometric morphism

Definition

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. The existential fibred site of f is the indexed functor $L_f : \mathcal{E}^{\text{op}} \to \text{Cat}$ sending any object E of $\mathcal{E}$ to the topos $\mathcal{F}/f^*(E)$ endowed with its canonical topology (for any arrow $k : E' \to E$ in $\mathcal{E}$, the pullback functor

$$L_f(k) := (f^*(k))^* : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$$

has a left adjoint

$$\exists_k : \mathcal{F}/f^*(E') \to \mathcal{F}/f^*(E)$$

given by composition with $f^*(k)$.

If $(\mathcal{C}, J)$ is a site of definition for $\mathcal{E}$, the composite of L_f with the canonical functor $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$ is also called the existential fibred site of f.

Remark

The existential fibred site $L_f : \mathcal{C}^{\text{op}} \to \text{Cat}$ of a geometric morphism $f : \mathcal{F} \to \mathbf{Sh}(\mathcal{C}, J)$ is J-reflecting, that is, it is a fibred site over $(\mathcal{C}, J)$.

Existential toposes

Theorem

Let $(\mathcal{C}, J)$ be a small-generated site and $L : \mathcal{C}^{\text{op}} \to \text{Cat}$ a fibred site over $\mathcal{C}$. Then L is existential if and only if the families on the category $\mathcal{G}(L)$ of the form

$$\{(e_i, \alpha_i) : (E_i, l_i) \to (E, l) \mid i \in I\}$$

where the family $\{\overline{\alpha_i} : \exists_{e_i}(l_i) \to l \mid i \in I\}$ is J^L_E-covering are the covering families for a Grothendieck topology J^{ext}_L, called the existential topology, on $\mathcal{G}(L)$.

Moreover, if L is an existential fibred site over $(\mathcal{C}, J)$, the existential topology J^{ext}_L contains the Giraud topology induced by J.

The relative topos

$$C_{p_L} : \mathbf{Sh}(\mathcal{G}(L), J^{\text{ext}}_L) \to \mathbf{Sh}(\mathcal{C}, J)$$

is called the existential topos of L.

Existential toposes

Proposition

- *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then, under the identification*

$$(1 \downarrow f^*) \cong \mathcal{G}(L_f),$$

  the topology J_f on $(1 \downarrow f^*)$, that is, the relative topology of f, corresponds to the existential topology $J^{\mathrm{ext}}_{L_f}$ on $\mathcal{G}(L_f)$, where L_f is the existential fibred site of f.

- *Every internal locale L in a topos $\mathcal{E}$ yields an existential fibred preorder site over the canonical site of $\mathcal{E}$.*

  Moreover, for any $E \in \mathcal{E}$, the topos of canonical sheaves on the locale $L(E)$ can be recovered as the localic reflection of the slice at E of the existential topos associated with L.

Morphisms of existential fibred sites

Definition

Given a topos $\mathcal{E}$ and existential fibred sites L and L' over $\mathcal{E}$, a morphism $\alpha : L \to L'$ is a morphism of indexed categories which is cartesian and cover-preserving at each fiber and which commutes with the left adjoints $\exists_e$ for any arrow e in $\mathcal{E}$.

Theorem

Given relative toposes $[f : \mathcal{F} \to \mathcal{E}]$ *and* $[f' : \mathcal{F}' \to \mathcal{E}]$*, the geometric morphisms* $f \to f'$ *over* $\mathcal{E}$ *correspond precisely to the morphisms of existential fibred sites* $L_{f'} \to L_f$*.*

Remark

This is a natural generalization of the classical result stating that the geometric morphisms $\mathbf{Sh}(L) \to \mathbf{Sh}(L')$ *correspond precisely to the frame homomorphisms* $L' \to L$*.*

Fibers of existential toposes

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site and L an existential fibred site over $(\mathcal{C}, J)$ and c an object of $\mathcal{C}$. Then the fibre $\mathbf{Sh}(\mathcal{G}(L), J_L^{\text{ext}})/C^*_{\pi_L}(l(c))$ *at c of the existential topos*

$$C_{\pi_L} : \mathbf{Sh}(\mathcal{G}(L), J_L^{\text{ext}}) \to \mathbf{Sh}(\mathcal{C}, J)$$

of L is equivalent to the topos of sheaves on the category $\mathcal{G}_c^{\text{ext}}(L)$ of elements of the functor $\mathrm{Hom}_{\mathcal{C}}(\pi_L(-), c)$, endowed with the Grothendieck topology $\widetilde{J_c}$ induced by J_L^{ext}.

For any arrow $k : c \to c'$ in $\mathcal{C}$, the pullback functor admits a left adjoint, given by the composition functor $\Sigma_{(C_{\pi_L})^*(l(k))}$ with $(C_{\pi_L})^*(l(k))$, which is induced by the comorphism of sites

$$E_k : \mathcal{G}_c^{\text{ext}}(L) \to \mathcal{G}_{c'}^{\text{ext}}(L)$$

given by composition with k.

Fibers of existential toposes

Proposition

For any object c of $\mathcal{C}$, the fiber at c of the existential topos of L is related to the topos of sheaves $\mathbf{Sh}(L(c), J^L_c)$ on the fiber of L at c via the hyperconnected *geometric morphism*

$$\mathbf{Sh}(i_c) \cong C_{ext_c} : \mathbf{Sh}(\mathcal{G}^{\text{ext}}_c(L), \widetilde{J}_c) \to \mathbf{Sh}(L(c), J^L_c)$$

induced respectively by the morphism of sites

$$i_c : (L(c), J^L_c) \to (\mathcal{G}^{\text{ext}}_c(L), \widetilde{J}_c)$$

sending an object x of $L(c)$ to the object $((c, x), 1_c)$ of $\mathcal{G}^{\text{ext}}_c(L)$, and by the (left adjoint) comorphism of sites

$$\text{ext}_c : (\mathcal{G}^{\text{ext}}_c(L), \widetilde{J}_c) \to (L(c), J^L_c)$$

sending an object $((d, y), f)$ of $\mathcal{G}^{\text{ext}}_c(L)$ to the object $\exists_f(y)$ of $L(c)$. Moreover, for any arrow $k : c \to c'$ in $\mathcal{C}$, the following diagram of comorphism of sites commutes:

$$\begin{array}{ccc} (\mathcal{G}^{\text{ext}}_c(L), \widetilde{J}_c) & \xrightarrow{\text{ext}_c} & (L(c), J^L_c) \\ \downarrow E_k & & \downarrow \exists_k \\ (\mathcal{G}^{\text{ext}}_{c'}(L), \widetilde{J}'_c) & \xrightarrow{\text{ext}_{c'}} & (L(c'), J^L_{c'}) \end{array}$$

Characterization of internal locales

The following corollary gives a characterization of internal locales in a topos $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on an arbitrary, not necessarily cartesian, small-generated site $(\mathcal{C}, J)$:

Corollary

Let $(\mathcal{C}, J)$ be a small-generated site. Then an internal locale in $\mathbf{Sh}(\mathcal{C}, J)$ is a functor $L : \mathcal{C}^{\text{op}} \to \text{Cat}$ taking values in the subcategory of frames and frame homomorphisms which is a J-sheaf and, when considered as a fibred site (by endowing each frame with its canonical topology), is existential i.e. the following conditions are satisfied:

(i) *Relative Beck-Chevalley condition: For any arrows $c : V \to Z$ and $d : W \to Z$ in $\mathcal{C}$ with common codomain and any $l \in L(V)$,*

$$L(d)(\exists_c(l)) = \bigvee_{(a,b)\in B_{(c,d)}} (\exists_b(L(a)(l))),$$

where $B_{(c,d)}$ is the collection of spans $(a : U \to V, b : U \to W)$ such that $c \circ a = d \circ c$;

(ii) *Frobenius reciprocity condition: for any $a : E \to E'$, $l \in L(E)$ and $l' \in L(E')$,*

$$\exists_a(L(a)(l') \wedge l) = \exists_a(l) \wedge l' \, .$$

Open fibred sites

Definition

We say that an existential fibred site $L : \mathcal{C}^{\text{op}} \to \text{Cat}$ is open if for every arrow $f : c \to c'$, the functor $\exists_f$ is cover-preserving.

Proposition

Let L be an open existential fibred site. Then, for any arrow $f : c \to c'$ in $\mathcal{C}$, the geometric morphism

$$\mathbf{Sh}(L(f)) \cong C_{\exists_f} : \mathbf{Sh}(L(c), J^L_c) \to \mathbf{Sh}(L(c'), J^L_{c'})$$

is open. Moreover, for any $c \in \mathcal{C}$, the geometric morphism

$$\mathbf{Sh}(i_c) \cong C_{ext_c} : \mathbf{Sh}(\mathcal{G}^{\text{ext}}_c(L), \widetilde{J}_c) \to \mathbf{Sh}(L(c), J^L_c)$$

of the above Proposition is open.

Remark

For any geometric morphism f, the existential fibred site L_f of f is open.

Applications to logic

The idea of investigating logical theories by using a fibrational formalism dates back to Lawvere and his notion of (hyper)doctrine. More specifically:

- A first-order theory $\mathbb{T}$ over a signature Σ is represented as a fibred preorder $L_{\mathbb{T}}$ indexed by the category Sort_{Σ} of sorts of Σ, whose objects are the finite list of variables of sorts in Σ and whose arrows are the maps between them which respect sorts.
- The indexed category $L_{\mathbb{T}}$ sends a context $\vec{x} = (x_1^{A_1}, \ldots, x_n^{A_n})$ to the poset $L_{\mathbb{T}}(\vec{x})$ of $\mathbb{T}$-provable equivalence classes of first-order formulas over Σ in the context $\vec{x}$.
- The transition functors are given by substitution, and they have adjoints on both sides, given by existential quantification and universal quantification.

Alternative syntactic sites

From a topos-theoretic point of view, if $\mathbb{T}$ is a geometric theory then:

- the presheaf topos $[\mathrm{Sort}_{\Sigma}^{\mathrm{op}}, \mathbf{Set}]$ is the classifying topos $\mathcal{E}_{\mathbb{O}_{\Sigma}}$ of the empty theory $\mathbb{O}_{\Sigma}$ consisting of just the sorts of Σ;
- (The geometric version of) $L_{\mathbb{T}}$ is an internal locale in $[\mathrm{Sort}_{\Sigma}^{\mathrm{op}}, \mathbf{Set}]$;
- $\mathbb{T}$ is a localic expansion of $\mathbb{O}_{\Sigma}$, whence the canonical geometric morphism $\mathcal{E}_{\mathbb{T}} \to \mathcal{E}_{\mathbb{O}_{\Sigma}}$ between their classifying toposes is localic.
- Hence the classifying topos $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$ identifies with the existential topos associated with the fibred site $L_{\mathbb{T}}$; in particular, we recover J. Wrigley's alternative syntactic site $(\mathcal{G}(L_{\mathbb{T}}), J^{\mathrm{ext}}_{L_{\mathbb{T}}})$.

As shown in the paper *Fibred sites and existential toposes*, many other alternative syntactic sites for the classifying topos of a theory can be obtained through the same method.

Completions of fibred preorder sites

It is possible to complete an arbitrary fibred preorder site to an internal locale:

Proposition

Let $(\mathbb{P}, K)$ be a fibered preordered site over a small-generated site $(\mathcal{C}, J)$. Then the canonical functor

$$\eta_{\mathbb{P}} : \mathbb{P} \to L_{C_{p_{\mathbb{P}}}},$$

where $L_{C_{p_{\mathbb{P}}}}$ is the internal locale associated with the geometric morphism $C_{p_{\mathbb{P}}}$, satisfies the universal property of the internal frame completion of $(\mathbb{P}, K)$.

It can be described as follows:

- *For any $c \in \mathcal{C}$, $L_{C_{p_{\mathbb{P}}}}(c)$ identifies with the frame*

$$\mathrm{ClSub}^{K}_{[\mathcal{G}(\mathbb{P})^{\mathrm{op}}, \mathbf{Set}]}(\mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}(-), c))$$

  of K-closed subobjects in $[\mathcal{G}(\mathbb{P})^{\mathrm{op}}, \mathbf{Set}]$ of the presheaf $\mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}(-), c)$.

Completions of fibred preorder sites

- The indexed functor $\eta_{\mathbb{P}}$ acts at an object $c \in \mathcal{C}$ as the functor

$$\eta_{\mathbb{P}}(c) : \mathbb{P}(c) \to L_{\mathcal{C}_{p_{\mathbb{P}}}}(c) = \mathrm{ClSub}^{K}_{[\mathcal{G}(\mathbb{P})^{\mathrm{op}},\mathbf{Set}]}(\mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}(-), c))$$

  sending any element $x \in \mathbb{P}(c)$ to the K-closure of the subfunctor of $\mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}(-), c)$ sending any object (c', x') of $\mathcal{G}(\mathbb{P})$ to the subset $S_{(c',x')} \subseteq \mathrm{Hom}_{\mathcal{C}}(p_{\mathbb{P}}((c', x')), c) = \mathrm{Hom}_{\mathcal{C}}(c', c)$ consisting of the arrows $g : c' \to c$ such that $x' \leq \mathbb{P}(g)(x)$.

Remarks

- *This generalizes the completion of a preorder site $(\mathcal{C}, J)$ to the frame $\mathrm{Id}_J(\mathcal{C})$ of J-ideals on $\mathcal{C}$.*
- *It would be interesting to investigate the connection between this kind of completions and the exact completions for Lawvere doctrines and the tripos-to-topos construction.*
- *More generally, the notion of existential fibred site should illuminate the relationships between Grothendieck toposes as built from sites and elementary toposes as built from triposes.*

For further reading

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