Toposes as unifying spaces: historical aspects and prospects

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The "crucial unifying notion" of topos

"It is the topos theme which is this "bed" or "deep river" where come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the "continuous" and that of "discontinuous" or discrete structures. It is what I have conceived of most broad to perceive with finesse, by the same language rich of geometric resonances, an "essence" which is common to situations most distant from each other coming from one region or another of the vast universe of mathematical things".

A. Grothendieck

Plan of the talk

- The notion of Grothendieck topos and its multifaceted nature
- The controversial reception of toposes
- Toposes as unifying 'bridges' : the underlying vision and a few examples
- Future perspectives

The multifaceted nature of toposes

The role of toposes as unifying spaces is intimately tied to their multifaceted nature.

For instance, a Grothendieck topos can be seen as :

- a generalized space
- a mathematical universe
- a theory modulo 'Morita-equivalence'

We shall now review each of these classical points of view, and then briefly discuss the more recent theory of topos-theoretic 'bridges', which combines all of them to provide tools for making toposes effective means for studying mathematical theories from multiple points of view, relating and unifying theories with each other and constructing 'bridges' across them.

Toposes as generalized spaces

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space X by a pair $(\mathscr{C}, J)$ consisting of a (small) category $\mathscr{C}$ and a 'generalized notion of covering' J on it, and taking sheaves (in a generalized sense) over the pair :

$$
\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, rightsquigarrow] & \mathbf{Sh}(X) \arrow[d, rightsquigarrow] \\
(\mathscr{C}, J) \arrow[r, dashed] & \mathbf{Sh}(\mathscr{C}, J)
\end{tikzcd}
$$

Topos-theoretic invariants

- The notion of a geometric morphism of toposes has notably allowed to build general comology theories starting from the categories of internal abelian groups or modules in toposes. In particular, the topos-theoretic viewpoint has allowed Grothendieck to refine and enrich the study of cohomology, up to the so-called 'six-operation formalism'.
  The cohomological invariants have had a tremendous impact on the development of modern Algebraic Geometry and beyond.

- On the other hand, also homotopy-theoretic invariants such as the fundamental group and the higher homotopy groups can be defined as invarants of toposes.

- Still, these are by no means the only invariants that one can consider on toposes : indeed, there are infinitely many invariants of toposes (of algebraic, logical, geometric or whatever nature), the notion of identity for toposes being simply categorical equivalence.

The "childish" idea of topos

"Like the very idea of sheaf (due to Leray), or that of scheme, like any "great idea" which upsets an inveterate vision of things, that of topos has something to disconcert by its naturality, or "evidence", by its simplicity (at the limit, one would say, by its innocence or simplistic character, or even its "stupidity"), by this particular quality which makes us cry out so often : "Oh, it is only that !", in a tone half-disappointed, half-envious, with in addition, perhaps, this implication of "eccentric", of "not serious", that one often reserves for all that confuses by an excess of unforeseen simplicity. To what comes to remind us, perhaps, of the long buried and denied days of our childhood..."

"On the other hand, I do not see anyone else on the mathematical scene, during the past three decades, who could have had this naivety, or this innocence, to take (in my place) this other crucial step among all, introducing the childish idea of topos (or even that of "sites")."

A. Grothendieck

Toposes as mathematical universes

A decade later, W. Lawvere and M. Tierney discovered that a topos could not only be seen as a generalized space, but also as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets (with the only important exception that one must argue constructively).

Amongst other things, this discovery made it possible to :

- Exploit the inherent 'flexibility' of the notion of topos to construct 'new mathematical worlds' having particular properties.
- Consider models of any kind of (first-order) mathematical theory not just in the classical set-theoretic setting, but inside every topos, and hence 'relativise' Mathematics.

Classifying toposes

The idea to consider toposes from the point of view of the structures that they classify dates back to the Ph.D. thesis "*Topos annelés et schemas rélatifs*" of Grothendieck's student M. Hakim, where four toposes relevant for algebraic geometry are characterized as the classifiers of certain kinds of rings.

On the other hand, Grothendieck talks in SGA 4 about classifying toposes of structures "which can be expressed in terms of finite projective limits and arbitrary inductive limits" and poses himself the problem of formalizing them :

[The exactness properties of the inverse image functor u^* of a geometric morphism of toposes $u : \mathscr{E} \to \mathscr{E}'$] ensure that for any kind of algebraic structure Σ whose data can be described in terms of arrows between the basic sets and of sets obtained from these by repeated applications of finite projective limits and arbitrary inductive limits, and for any "object of $\mathscr{E}'$ endowed with a structure of type Σ", its image under u^* is endowed with the same kind of structure. Rather than entering the uninviting task of giving a precise meaning to this statement and of justifying it formally, we advise the reader to make it explicit and to get convinced of its validity for species of structures such as that of group, ring, module over a ring, comodule over a ring, bialgebra over a ring, torsor for a group.

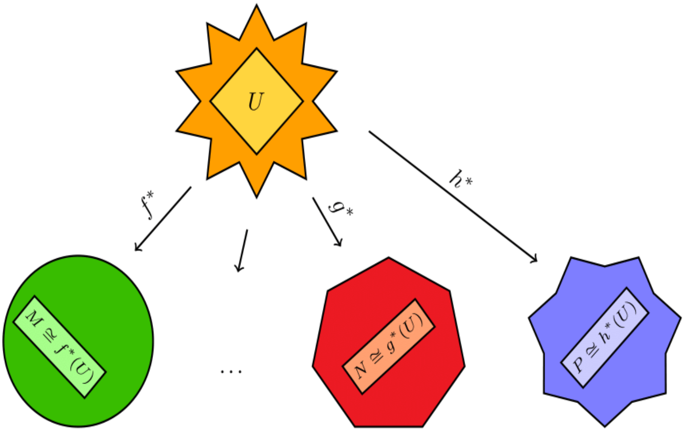
Toposes as theories modulo 'Morita equivalence'

Thanks to the work of several categorical logicians, notably including W. Lawvere, G. Reyes, A. Joyal, M. Makkai, J. Bénabou and J. Cole, in the seventies, the "geometric logic" invoked by Grothendieck was defined and it was shown that :

- To any geometric (first-order) theory $\mathbb{T}$ one can canonically associate a Grothendieck topos $\mathscr{E}_{\mathbb{T}}$, called its classifying topos, which represents its 'semantical core'.
- The topos $\mathscr{E}_{\mathbb{T}}$ is characterized by the following universal property : for any Grothendieck topos $\mathscr{E}$, we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

  natural in $\mathscr{E}$, where $\mathbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}})$ is the category of geometric morphisms $\mathscr{E} \to \mathscr{E}_{\mathbb{T}}$ and $\mathbb{T}\text{-mod}(\mathscr{E})$ is the category of models of $\mathbb{T}$ in $\mathscr{E}$.



Classifying topos

Toposes as theories up to 'Morita-equivalence'

- Two mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is if and only if they are indistinguishable from a semantic point of view ; such theories are said to be Morita-equivalent.

- Conversely, every Grothendieck topos arises as the classifying topos of some theory.

- So a topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.

The thwarted reception of toposes

Grothendieck repeatedly complains in *Récoltes et Semailles* about the negative reception of toposes in the mathematical community, which he attributes primarily to the lack of vision of his former colleagues. He writes for example :

"I learned little by little, I cannot say enough how, that several notions which were part of the forgotten vision, had not only fallen into disuse, but had become, in a certain circle of fine people, the object of a condescending disdain. This was the case, in particular, for the crucial unifying notion of topos, at the very heart of the new geometry - the very one which provides the common geometric intuition for topology, algebraic geometry and arithmetic - that which also allowed me to introduce both the étale and ℓ-adic cohomological tool, and the main ideas (more or less forgotten since, it is true...) of crystalline cohomology. To tell the truth, it was my very name, over the years, which insidiously, mysteriously, had become an object of derision - as a synonym for muddy endless spooling (such as those on those famous "toposes", indeed, or these "motives" which fold back the ears and which nobody had ever seen ...), of hair cut in four to the length of a thousand pages, and of bloated and gigantic chatter on things which, in any case, everyone has always known and without having expected them..."

The thwarted reception of toposes

"For fifteen years (since my departure from the mathematical scene), the fruitful unifying idea and the powerful tool of discovery which is the notion of topos, is maintained by a certain circle banished from the notions deemed to be serious. Few of the topologists today still have the slightest suspicion of this considerable potential expansion of their science, and of the new resources it offers."

"Given the disdain with which some of my former students (...) have taken pleasure in treating this crucial unifying notion, the latter has been condemned since my departure to a marginal existence. (...) toposes (...) are nevertheless encountered at every step in geometry - but we can of course very well do without seeing them, as people have avoided for millennia to see groups of symmetries, sets, or the number zero."

The vision, and the tool

*"The set of two consecutive seminars SGA 4 and SGA 5 (which for me are just a **single** "seminar") develops from nothing, both the powerful instrument of synthesis and discovery represented by the **language** of toposes, and the tool, perfectly developed, of perfect effectiveness, that is étale cohomology - better understood in its essential formal properties, from that moment on, than even the cohomological theory of ordinary spaces."*

"These two seminars are for me inseparably linked. They represent, in their unity, both the vision, and the tool - toposes, and a complete formalism of étale cohomology. While the vision is still rejected today, the tool has, throughout more than twenty years, deeply renewed algebraic geometry in its most fascinating aspect for me of all - the "arithmetic" aspect, apprehended by an intuition, and by a conceptual and technical baggage, of "geometric" nature."

*"The operation "Étale cohomology" consisted in **discrediting the unifying vision** of toposes (such as "nonsense", spooling etc.) ... and on the other hand, to **appropriate the tool**, i.e. the paternity of the ideas, techniques and results that I had developed on the theme of étale cohomology."*

Elementary toposes

"For almost fifteen years, it has been part of the bon ton in the "big world", to look down on anyone who dares to pronounce the word "topos", unless it is for a joke or he has the excuse of being a logician. (These are people known to be like no other and to whom we must forgive certain whims...)"

In fact, categorical logicians too, as well as geometers, after defining geometric logic during the seventies, have essentially abandoned the study of Grothendieck toposes as classifiers of geometric theories, in order to work on other themes such as that of "elementary toposes" of W. Lawvere and M. Tierney, a kind of category which differs from Grothendieck toposes notably by the fact of being finely axiomatizable in the language of categories but of not having all colimits nor of being always representable by sites.

Sites without toposes, toposes without sites

- As we said above, most algebraic geometers after Grothendieck essentially abandoned the notion of topos by concentrating on the study of particular cohomological theories associated with specific geometric sites, probably for the sake of pragmatism. This practice of neglecting toposes in favor of sites - which could be summed up by the slogan **"sites without toposes"** - has been largely shared within this community.
- On the other hand, the choice of most categorical logicians of neglecting Grothendieck toposes in favour of "elementary toposes" has led them to study toposes without reference to their presentations, an approach which we could sum up by the slogan **"toposes without sites"**. This choice was actually based on a bias rejecting both infinitary and higher-order constructions.
- Remarkably, what has been missing in both schools is the **integration** between the “concrete” level of sites and the “abstract” or “metamathematical” level of toposes, an integration which is the essential condition for a fruitful use of toposes as unifying spaces in mathematics. Indeed, as we shall see, this requires working at two levels, which must not be confused nor cut off from one another.

Toposes as unifying 'bridges'

Since the times of my Ph.D. studies, I have developed a theory and a number of techniques which allow to effectively use Grothendieck toposes as unifying spaces in mathematics, thus vindicating Grothendieck's aspirations as to the central role of his notion of topos.

This theory, introduced in the programmatic paper "*The unification of Mathematics via Topos Theory*" in 2010, provides means for exploiting the technical flexibility inherent to the concept of topos - more precisely, the possibility of presenting toposes by a multiplicity of different ways, to build unifying 'bridges' across different mathematical theories having an equivalent, or strictly related, semantic content.

These techniques have already generated several deep applications in different fields of mathematics ; still, the potential of this theory has just started to be exploited.

In fact, these 'bridges' have proved useful not only for relating different mathematical theories with each other, but also for studying a given mathematical theory within a specific domain in a dynamical, interdisciplinary way.

Toposes as *bridges*

- In the topos-theoretic study of theories, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them).
- The existence of theories which are Morita-equivalent to each other translates into the existence of different sites of definition (or, more generally, presentations) for the same Grothendieck topos.
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across different Morita-equivalent theories :

$$\begin{tikzcd}
 & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed] & \\
\mathbb{T} \arrow[ur, dashed] & & \mathbb{T}'
\end{tikzcd}$$

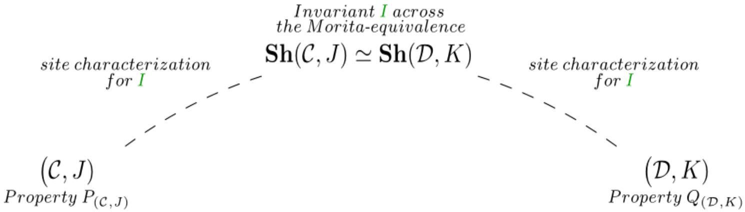
- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

The 'bridge-building' technique

More precisely :

- Decks of 'bridges' : Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges' : Site characterizations for topos-theoretic invariants (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)

For example :



This 'bridge' yields a logical equivalence (or an implication) between the 'concrete' properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

A few selected applications

Since this theory was introduced, several applications of it have been obtained in different fields of Mathematics, such as :

- Model theory (topos-theoretic Fraïssé theorem)
- Proof theory (various results for first-order theories)
- Algebra (topos-theoretic generalization of topological Galois theory)
- Topology (topos-theoretic interpretation/generation of Stone-type and Priestley-type dualities)
- Functional analysis (various results on Gelfand spectra and Wallman compactifications)
- Many-valued logics and lattice-ordered groups (two joint papers with A. C. Russo)
- Cyclic homology, as reinterpreted by A. Connes (work on *"cyclic theories"*, jointly with N. Wentzlaff)
- Algebraic geometry (logical analysis of (co)homological motives, cf. the paper *"Syntactic categories for Nori motives"* joint with L. Barbieri-Viale and L. Lafforgue)

Topological Galois theory as a 'bridge'

Theorem

Let $\mathscr{C}$ be a small category satisfying the amalgamation and joint embedding properties, et let u be a $\mathscr{C}$-universal et $\mathscr{C}$-ultrahomogeneous object of the ind-completion Ind-$\mathscr{C}$ *of $\mathscr{C}$. Then there is an equivalence of toposes*

$$\mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)),$$

where $Aut(u)$ is endowed with the topology in which a basis of open neighbourhoods of the identity is given by the subgroups of the form $I_\chi = \{\alpha \in Aut(u) \mid \alpha \circ \chi = \chi\}$ for $\chi : c \to u$ an arrow in Ind-$\mathscr{C}$ *from an object c of $\mathscr{C}$.*
This equivalence is induced by the functor

$$F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cont}(Aut(u))$$

which sends any object c of $\mathscr{C}$ on the set $\mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{C}}(c, u)$ (endowed with the obvious action of $Aut(u)$) and any arrow $f : c \to d$ in $\mathscr{C}$ to the $Aut(u)$-equivariant map

$$- \circ f : \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{C}}(d, u) \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathscr{C}}(c, u)\ .$$

Topological Galois theory as a 'bridge'

The following result arises from two 'bridges', respectively obtained by considering the invariant notions of atom and of arrow between atoms.

Theorem

Under the hypotheses of the last theorem, the functor F is full and faithful if and only if every arrow of $\mathscr{C}$ is a strict monomorphism, and it is an equivalence on the full subcategory $\mathbf{Cont}_t(Aut(u))$ of $\mathbf{Cont}(Aut(u))$ on the non-empty transitive actions if $\mathscr{C}$ is moreover atomically complete.

$$\begin{tikzcd} & \mathbf{Sh}(\mathscr{C}^{\mathrm{op}}, J_{at}) \simeq \mathbf{Cont}(Aut(u)) \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathscr{C}^{\mathrm{op}} & & \mathbf{Cont}_t(Aut(u)) \end{tikzcd}$$

This theorem generalizes Grothendieck's theory of Galois categories and can be applied for generating Galois-type theories in different fields of Mathematics, for example that of finite groups and that of finite graphs.

Moreover, if a category $\mathscr{C}$ satisfies the first but not the second condition of the theorem, our topos-theoretic approach gives us a fully explicit way to complete it, by means of the addition of 'imaginaries', so that also the second condition gets satisfied.

Stone-type dualities through 'bridges'

The 'bridge-building' technique allows one to unify all the classical Stone-type dualities between special kinds of preorders and partial orders, locales or topological spaces as instances of just one topos-theoretic phenomenon, and to generate many new such dualities.

More precisely, this machinery generates Stone-type dualities/equivalences by functorializing 'bridges' of the form

$$\begin{array}{ccc} & \mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}}) \simeq \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}}) & \\ \mathscr{C} & & \mathscr{D} \end{array}$$

where

- $\mathscr{C}$ is a preorder (regarded as a category),
- $J_{\mathscr{C}}$ is a (subcanonical) Grothendieck topology on $\mathscr{C}$,
- $\mathscr{C}$ is a $K_{\mathscr{D}}$-dense full subcategory of $\mathscr{D}$, and
- $J_{\mathscr{C}}$ is the induced Grothendieck topology $(K_{\mathscr{D}})|_{\mathscr{C}}$ on $\mathscr{C}$.

Stone-type dualities through 'bridges'

Our machinery relies on the following key points :

- The possibility of defining Grothendieck topologies on posets in an intrinsic way which exploits the lattice-theoretic structure present on them.
- The possibility of functorializing the assignments $\mathscr{C} \to \mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$ and $\mathscr{D} \to \mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$ by means of morphisms or comorphisms of sites.
- The possibility of recovering (under suitable hypotheses which are satisfied in a great number of cases) a given preordered structure from the associated topos by means of a topos-theoretic invariant.
  More precisely, if the topologies $K_{\mathscr{D}}$ (resp. $J_{\mathscr{C}}$) can be 'uniformly described through an invariant C of families of subterminals in a topos' then the elements of $\mathscr{D}$ (resp. of $\mathscr{C}$) can be recovered as the subterminal objects of the topos $\mathbf{Sh}(\mathscr{D}, K_{\mathscr{D}})$ (resp. $\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$) which satisfy a condition of C-compactness.

Future directions

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

In the next years, we intend to continue pursuing the development of these general unifying methodologies both at the theoretical level and at the applied level, in order to continue developing the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as key concepts defining a new way of doing Mathematics liable to bring distinctly new insights in a great number of different subjects.

Future directions

Central themes in this programme will be :

- investigation of important dualities or correspondences in Mathematics from a topos-theoretic perspective (in particular, the theory of motives, class field theory and the Langlands programme)
- systematic study of invariants of toposes in terms of their presentations, and introduction of new invariants which capture important aspects of concrete mathematical problems
- interpretation and generalization of important parts of classical and modern model theory in terms of toposes and development of a functorial model theory
- introduction of new methodologies for generating Morita-equivalences
- development of general techniques for building spectra by using classifying toposes
- generalization of the 'bridge' technique to the setting of higher categories and toposes through the introduction of higher geometric logic
- development of a relative theory of classifying toposes

For further reading

O. Caramello,
La « notion unificatrice » de topos,
to appear in the proceedings volume of the «Lectures Grothendieckiennes», École Normale Supérieure, Paris.

O. Caramello
Grothendieck toposes as unifying 'bridges' in Mathematics,
Mémoire d'habilitation à diriger des recherches,
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available from my website www.oliviacaramello.com.

O. Caramello
Theories, Sites, Toposes : Relating and studying mathematical theories through topos-theoretic 'bridges',
Oxford University Press (2017).