The unification of Mathematics
via
Topos Theory

Olivia Caramello

University of Cambridge

Toposes as unifying spaces in Mathematics

In this lecture, whenever I use the word 'topos', I really mean 'Grothendieck topos'.

Recall that a Grothendieck topos can be seen as:

- a generalized space
- a mathematical universe
- a geometric theory modulo 'Morita-equivalence' where 'Morita-equivalence' is the equivalence relation which identifies two (geometric) theories precisely when they have equivalent categories of models in any Grothendieck topos $\mathscr{E}$, naturally in $\mathscr{E}$.

In this talk, we present a new view of toposes as unifying spaces which can serve as bridges for transferring information, ideas and results between distinct mathematical theories. This approach, first introduced in my Ph.D. thesis, has already generated ramifications into distinct mathematical fields and points towards a realization of Topos Theory as a unifying theory of Mathematics.

Some examples from my research

- Model Theory (a topos-theoretic interpretation of Fraïssé's construction in Model Theory)
- Algebra (an application of De Morgan's law to the theory of fields - jointly with P. T. Johnstone)
- Topology (a unified approach to Stone-type dualities)
- Proof Theory (an equivalence between the traditional proof system of geometric logic and a categorical system based on the notion of Grothendieck topology)
- Definability (applications of universal models to definability)

These are just a few examples selected from my research; as I see it, the interest of them especially lies in the fact that they demonstrate the technical usefulness and centrality of the philosophy '*Toposes as bridges*' described below: without much effort, one can generate an infinite number of new theorems by applying these methodologies!

Geometric theories

Definition

- A geometric formula over a signature Σ is any formula (with a finite number of free variables) built from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.
- A geometric theory over a signature Σ is any theory whose axioms are of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are geometric formulae over Σ and $\vec{x}$ is a context suitable for both of them.

Fact

Most of the theories naturally arising in Mathematics are geometric; and if a finitary first-order theory is not geometric, we can always associate to it a finitary geometric theory over a larger signature (the so-called Morleyization of the theory) with essentially the same models in the category **Set** *of sets.*

The notion of classifying topos

Definition

Let $\mathbb{T}$ be a geometric theory over a given signature. A classifying topos of $\mathbb{T}$ is a Grothendieck topos $\mathbf{Set}[\mathbb{T}]$ such that for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

natural in $\mathscr{E}$.

Theorem

Every geometric theory (over a given signature) has a classifying topos. Conversely, every Grothendieck topos arises as the classifying topos of some geometric theory.

The classifying topos of a geometric theory $\mathbb{T}$ can always be constructed canonically from the theory by means of a syntactic construction, namely as the topos of sheaves $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ on the geometric syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ with respect to the syntactic topology $J_{\mathbb{T}}$ on it (i.e. the canonical Grothendieck topology on $\mathscr{C}_{\mathbb{T}}$).

Toposes as bridges I

Definition

- By a site of definition of a Grothendieck topos $\mathscr{E}$, we mean a site $(\mathscr{C}, J)$ such that $\mathscr{E} \simeq \mathbf{Sh}(\mathscr{C}, J)$.
- We shall say that two geometric theories are Morita-equivalent if they have equivalent categories of models in every Grothendieck topos $\mathscr{E}$, naturally in $\mathscr{E}$.

Note that 'to be Morita-equivalent to each other' defines an equivalence relation of the collection of all geometric theories.

- Two geometric theories are Morita-equivalent if and only if they are biinterpretable in each other (in a generalized sense).
- On the other hand, the notion of Morita-equivalence is a 'semantical' one, and we can expect most of the categorical equivalences between categories of models of geometric theories in **Set** to 'lift' to Morita-equivalences.

Toposes as bridges II

- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences (either as Morita-equivalences themselves or as arising from the process of 'functorializing' Morita-equivalences).
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- It is fair to say that the notion of Morita-equivalence formalizes in many situations the feeling of 'looking at the same thing in different ways, which explains why it is ubiquitous in Mathematics.
- Also, the notion captures the intrinsic dynamism inherent to the notion of mathematical theory; indeed, a mathematical theory alone gives rise to an infinite number of Morita-equivalences.

Toposes as bridges III

- Two geometric theories have equivalent classifying toposes if and only if they are Morita-equivalent to each other.
- Hence, a topos can be seen as a *canonical representative* of equivalence classes of geometric theories modulo Morita-equivalence. So, we can think of a topos as embodying the 'common features' of mathematical theories which are Morita-equivalent to each other.
- The essential features of Morita-equivalences are all 'hidden' inside Grothendieck toposes, and can be revealed by using their different sites of definition.
    - *Conceptually*, a property of geometric theories which is stable under Morita-equivalence *is* a property of their classifying topos.
    - *Technically*, considered the richness and flexibility of topos-theoretic methods, we can expect these properties to be expressible as topos-theoretic invariants (in the sense of properties of toposes written in the language of Topos Theory).

Toposes as bridges IV

- The underlying intuition behind this is that a given mathematical property can manifest itself in several different forms in the context of mathematical theories which have a common 'semantical core' but a different linguistic presentation.
- The remarkable fact is that if the property is formulated as a topos-theoretic invariant on some topos then the expression of it in terms of the different theories classified by the topos is determined to a great extent by the structural relationship between the topos and the different sites of definition for it.
- Indeed, the fact that different mathematical theories have equivalent classifying toposes translates into the existence of different sites of definition for one topos.
- Topos-theoretic invariants can then be used to transfer properties from one theory to another.
- In fact, any invariant behaves like a 'pair of glasses' which allows us to discern certain information which is 'hidden' in the given Morita-equivalence. Different invariants enable us to extract different information.

Toposes as bridges V

- This methodology is technically feasible because the relationship between a site $(\mathscr{C}, J)$ and the topos $\mathbf{Sh}(\mathscr{C}, J)$ which it 'generates' is often very natural, enabling us to easily transfer invariants across different sites.
- A topos thus acts as a 'bridge' which allows the transfer of information and results between theories which are Morita-equivalent to each other:

$$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{C}', J')$$

$$(\mathscr{C}, J) \qquad\qquad\qquad\qquad (\mathscr{C}', J')$$

Toposes as bridges VI

- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories. Indeed, as shown in my thesis, important topos-theoretic invariants considered on the classifying topos $\mathbf{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$ translate into interesting logical (i.e. syntactic or semantic) properties of $\mathbb{T}$.
- The fact that topos-theoretic invariants often specialize to important properties or constructions of natural mathematical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole.

A new way of doing Mathematics I

- The 'working mathematician' could very well attempt to formulate his or her properties of interest in terms of topos-theoretic invariants, and derive equivalent versions of them by using alternative sites.
- Also, whenever one discovers a duality or an equivalence of mathematical theories, one should try to interpret it in terms of a Morita-equivalence, calculate the classifying topos of the two theories and apply topos-theoretic methods to extract new information about it.
- These methodologies define a new way of doing Mathematics which is 'upside-down' compared with the 'usual' one. Instead of starting with simple ingredients and combining them to build more complicated structures, one assumes as primitive ingredients rich and sophisticated mathematical entities, namely Morita-equivalences and topos-theoretic invariants, and extracts from them a huge amount of information relevant for classical mathematics.

A new way of doing Mathematics I

- The 'working mathematician' could very well attempt to formulate his or her properties of interest in terms of topos-theoretic invariants, and derive equivalent versions of them by using alternative sites.
- Also, whenever one discovers a duality or an equivalence of mathematical theories, one should try to interpret it in terms of a Morita-equivalence, calculate the classifying topos of the two theories and apply topos-theoretic methods to extract new information about it.
- These methodologies define a new way of doing Mathematics which is 'upside-down' compared with the 'usual' one. Instead of starting with simple ingredients and combining them to build more complicated structures, one assumes as primitive ingredients rich and sophisticated mathematical entities, namely Morita-equivalences and topos-theoretic invariants, and extracts from them a huge amount of information relevant for classical mathematics.

A new way of doing Mathematics II

- There is an strong element of automatism in these techniques; by means of them, one can generate new mathematical results without really making any creative effort: indeed, in many cases one can just readily apply the general characterizations connecting properties of sites and topos-theoretic invariants to the particular Morita-equivalence under consideration.
- The results generated in this way are in general non-trivial; in some cases they can be rather 'weird' according to the usual mathematical standards (although they might still be quite deep) but, with a careful choice of Morita-equivalences and invariants, one can easily get interesting and natural mathematical results.
- In fact, a lot of information that is not visible with the usual 'glasses' is revealed by the application of this machinery.
- On the other hand, the range of applicability of these methods is boundless within Mathematics, by the very generality of the notion of topos.

Topos-theoretic invariants

Definition

By a topos-theoretic invariant we mean a property of (or a construction involving) toposes which is stable under categorical equivalence.

Examples of topos-theoretic invariants include:

- to be a classifying topos for a geometric theory.
- to be a Boolean (resp. De Morgan) topos.
- to be an atomic topos.
- to be equivalent to a presheaf topos.
- to be a connected (resp. locally connected, compact) topos.
- to be a subtopos (in the sense of geometric inclusion) of a given topos.
- to have enough points.
- to be two-valued.

Of course, there are a great many others, and one can always introduce new ones!

Sometimes, in order to obtain more specific results, it is convenient to consider invariants of objects of toposes rather than 'global' invariants of toposes.

Subtoposes

Definition

A subtopos of a topos $\mathscr{E}$ is a geometric inclusion of the form $\mathbf{sh}_j(\mathscr{E}) \hookrightarrow \mathscr{E}$ for a local operator j on $\mathscr{E}$.

Fact

- *A subtopos of a topos $\mathscr{E}$ can be thought of as an equivalence class of geometric inclusions with codomain $\mathscr{E}$; hence, the notion of subtopos is a topos-theoretic invariant.*
- *If $\mathscr{E}$ is the topos $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a site $(\mathscr{C}, J)$, the subtoposes of $\mathscr{E}$ are in bijective correspondence with the Grothendieck topologies J' on $\mathscr{C}$ which contain J (i.e. such that every J-covering sieve is J'-covering).*

The duality theorem

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. A quotient of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ over Σ such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$.
- Let $\mathbb{T}$ and $\mathbb{T}'$ be geometric theories over a signature Σ. We say that $\mathbb{T}$ and $\mathbb{T}'$ are syntactically equivalent, and we write $\mathbb{T} \equiv_s \mathbb{T}'$, if for every geometric sequent σ over Σ, σ is provable in $\mathbb{T}$ if and only if σ is provable in $\mathbb{T}'$.

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the $\equiv_s$-equivalence classes of quotients of $\mathbb{T}$ and the subtoposes of the classifying topos $\mathbf{Set}[\mathbb{T}]$ of $\mathbb{T}$.

A simple example

In light of the fact that the notion of subtopos is a topos-theoretic invariant, the duality theorem allows us to easily transfer information between quotients of geometric theories classified by the same topos.

For example, consider the following problem. Suppose to have a Morita-equivalence between two geometric theories $\mathbb{T}$ and $\mathbb{S}$.

Question: If $\mathbb{T}'$ is a quotient of $\mathbb{T}$, is there a quotient $\mathbb{S}'$ of $\mathbb{S}$ such that the given Morita-equivalence restricts to a Morita-equivalence between $\mathbb{T}'$ and $\mathbb{S}'$?

The duality theorem gives a straight positive answer to this question. In fact, both quotients of $\mathbb{T}$ and quotients of $\mathbb{S}$ correspond bijectively with subtoposes of the classifying topos $\mathbf{Set}[\mathbb{T}] = \mathbf{Set}[\mathbb{S}]$.

Note the role of the classifying topos as a 'bridge' between the two theories!

A simple example

In light of the fact that the notion of subtopos is a topos-theoretic invariant, the duality theorem allows us to easily transfer information between quotients of geometric theories classified by the same topos.
For example, consider the following problem. Suppose to have a Morita-equivalence between two geometric theories $\mathbb{T}$ and $\mathbb{S}$.

Question: If $\mathbb{T}'$ is a quotient of $\mathbb{T}$, is there a quotient $\mathbb{S}'$ of $\mathbb{S}$ such that the given Morita-equivalence restricts to a Morita-equivalence between $\mathbb{T}'$ and $\mathbb{S}'$?

The duality theorem gives a straight positive answer to this question. In fact, both quotients of $\mathbb{T}$ and quotients of $\mathbb{S}$ correspond bijectively with subtoposes of the classifying topos $\textbf{Set}[\mathbb{T}] = \textbf{Set}[\mathbb{S}]$.

Note the role of the classifying topos as a 'bridge' between the two theories!

Applications

The duality theorem realizes a unification between the theory of elementary toposes and geometric logic. In fact, there are several notions and results in elementary Topos Theory which apply to subtoposes of a given topos; all of these results can then be transferred into the world of Logic by passing through the theory of Grothendieck toposes. Examples include:

- The notion of Booleanization (resp. DeMorganization) of a geometric theory.
- The logical interpretation of the surjection-inclusion factorization of a geometric morphism.
- The Heyting algebra structure on the collection of (syntactic-equivalence classes of) geometric theories over a given signature.

Theories of presheaf type

Definition

- A geometric theory $\mathbb{T}$ over a signature Σ is said to be of presheaf type if it is classified by a presheaf type.
- A model M of a theory of presheaf type $\mathbb{T}$ in the category **Set** is said to be finitely presentable if the functor $Hom_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M,-) : \mathbb{T}\text{-mod}(\mathbf{Set}) \to \mathbf{Set}$ preserves filtered colimits.

The class of theories of presheaf type contains all the finitary algebraic theories and many other significant mathematical theories.

Fact

For any theory of presheaf type $\mathscr{C}$, we have two different representations of its classifying topos:

$$[f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

where $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ is the category of finitely presentable $\mathbb{T}$-models.

Note that this gives us a perfect opportunity to test the effectiveness of the philosophy 'toposes as bridges'!

Topos-theoretic Fraïssé's construction

Theorem

*Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) satisfies both amalgamation and joint embedding properties. Then any two countable homogeneous $\mathbb{T}$-models in **Set** are isomorphic.*

The quotient of $\mathbb{T}$ axiomatizing the homogeneous $\mathbb{T}$-models is precisely the Booleanization of $\mathbb{T}$.

The theorem is proved by means of an investigation of topos $\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$

- geometrically
- syntactically
- semantically

The main result arises from an integration of these three lines of investigation according to the principles 'toposes as bridges' explained above.

For further reading

- O. Caramello.
  The unification of Mathematics via Topos Theory,
  arXiv:math.CT/1006.3930, 2010
- O. Caramello.
  The Duality between Grothendieck toposes and Geometric Theories, Ph.D. thesis
  University of Cambridge, 2009