Toposes as *bridges* for unifying Mathematics

Olivia Caramello

Unifying Mathematics

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Toposes as *bridges* for unifying Mathematics

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Unifying Mathematics

- Set Theory has represented the first serious attempt of Logic to unify Mathematics at least at the level of language.
- Later, Category Theory offered an alternative abstract language in which most of Mathematics can be formulated.

Anyway, both these systems realize a unification which is still limited in scope, in the sense that, even though each of them provides a way of expressing and organizing Mathematics in one single language, they do not offer by themselves effective methods for an actual transfer of knowledge between distinct fields.

Instead, the methodologies introduced in the paper

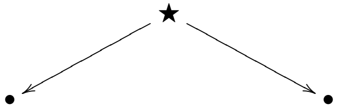
The unification of Mathematics via Topos Theory

define a different and more substantial approach to the problem of 'unifying Mathematics'.

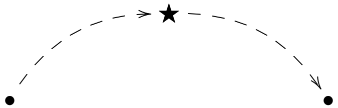
The concept of unification

We can distinguish between two different kinds of unification.

- ‘Static’ unification (through generalization): two concepts are seen to be special instances of a more general one:



- ‘Dynamic’ unification (through construction): two objects are related to each other through a third one (usually constructed from each of them), which acts like a ‘bridge’ enabling a transfer of information between them.



The transfer of information arises from the process of ‘translating’ properties (resp. constructions) on the ‘bridge object’ into properties (resp. constructions) on the two objects.

The eclectic nature of toposes

In this lecture, whenever I use the word 'topos', I really mean 'Grothendieck topos'.

A Grothendieck topos can be seen as:

- a generalized space
- a mathematical universe
- a theory modulo 'Morita-equivalence'

In this lecture, I will describe the fundamental principles underlying a new view of toposes as unifying spaces which can serve as bridges for transferring information, ideas and results between distinct mathematical theories.

This approach, first introduced in my Ph.D. thesis, has already generated a great number of non-trivial applications into distinct mathematical fields, and points towards a realization of Topos Theory as a unifying theory of Mathematics.

Some examples from my research

- Model Theory (a topos-theoretic interpretation of Fraïssé's construction in Model Theory)
- Algebra (an application of De Morgan's law to the theory of fields - jointly with P. T. Johnstone)
- Topology (a unified approach to Stone-type dualities)
- Proof Theory (an equivalence between the traditional proof system of geometric logic and a categorical system based on the notion of Grothendieck topology)
- Definability (applications of universal models to definability)

These are just a few examples selected from my research. As I see it, their interest especially lies in the fact that they demonstrate the technical usefulness and centrality of the philosophy '*Toposes as bridges*' described below: without much effort, one can generate an infinite number of new results by applying these methodologies!

Toposes as generalized spaces

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space X by a pair $(\mathscr{C}, J)$ consisting of a (small) category $\mathscr{C}$ and a 'generalized notion of covering' J on it, and taking sheaves (in a generalized sense) over the pair:

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{Sh}(X) \\ \rightsquigarrow & & \rightsquigarrow \\ (\mathscr{C}, J) & \dashrightarrow & \mathbf{Sh}(\mathscr{C}, J) \end{array}$$

Toposes as mathematical universes

A decade later, W. Lawvere and M. Tierney discovered that a topos could not only be seen as a generalized space, but also as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets (with the only exception that one must argue constructively).

Amongst other things, this discovery made it possible to:

- Exploit the inherent ‘flexibility’ of the notion of topos to construct ‘new mathematical worlds’ having particular properties.
- Consider models of any kind of (first-order) mathematical theory not just in the classical set-theoretic setting, but inside every topos, and hence ‘relativise’ Mathematics.

Toposes as theories

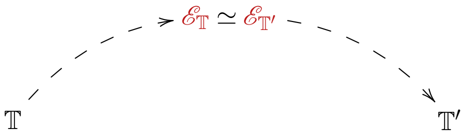
- To any mathematical theory $\mathbb{T}$ (of a general specified form) one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called the classifying topos of the theory, which represents its 'semantical core'.
- Two mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is if and only if they are indistinguishable from a semantic point of view; such theories are said to be Morita-equivalent.
- Conversely, every Grothendieck topos arises as the classifying topos of some theory.
- So a topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.

Toposes as *bridges* I

- The notion of Morita-equivalence formalizes in many situations the feeling of 'looking at the same thing in different ways', which explains why it is ubiquitous in Mathematics.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- Moreover, the notion of Morita-equivalence captures the intrinsic dynamism inherent to the notion of mathematical theory; indeed, a mathematical theory alone gives rise to an infinite number of Morita-equivalences.

Toposes as *bridges* II

- The existence of different theories with the same classifying topos translates, at the technical level, into the existence of different representations for the same topos.
- Topos-theoretic invariants can thus be used to transfer information from one theory to another:



- The transfer of information takes place by expressing a given invariant in terms of the different representation of the topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

Toposes as *bridges* III

- These methodologies are technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- As a matter of fact, these methods define a new way of doing Mathematics which is 'upside-down' compared with the 'usual' one. Indeed, one assumes as primitive ingredients abstract notions such as Morita-equivalences and topos-theoretic invariants, and proceeds to extract from them concrete information relevant for classical mathematics.
- Moreover, there is an strong element of automatism in these techniques; by means of them, one can generate new mathematical results without really making any creative effort. Indeed, in many cases one can just readily apply the general characterizations expressing topos-theoretic invariants on a topos in terms of properties of its representations to the particular Morita-equivalence under consideration.

Why toposes?

- Generality: Unlike most of the invariants used in Mathematics, the level of generality of topos-theoretic invariants is such to make them suitable for comparing with each other (first-order) mathematical theories of essentially any kind.
- Expressivity: On the other hand, many important invariants arising in Mathematics can be expressed as topos-theoretic invariants.
- Centrality: The fact that topos-theoretic invariants often specialize to important properties or constructions of natural mathematical or logical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole.
- Technical flexibility: Toposes are mathematical universes which are very rich in terms of internal structure; moreover, they have a very-well behaved representation theory, which makes them extremely effective computationally when considered as 'bridges'.

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Structural translations

The method of bridges can be interpreted linguistically as a methodology for translating concepts from one context to another.

But which kind of translation is this?

In general, we can distinguish between two essentially different approaches to translation.

- The 'dictionary-oriented' or 'bottom-up' approach, consisting in a dictionary-based renaming of the single words composing the sentences.
- The 'invariant-oriented' or 'top-down' approach, consisting in the identification of appropriate concepts that should remain invariant in the translation, and in the subsequent analysis of how these invariants can be expressed in the two languages.

The topos-theoretic translations are of the latter kind. Indeed, the invariant properties are topos-theoretic invariants defined on toposes, and the expression of these invariants in terms of the two different theories is essentially determined by the structural relationship between the topos and its two different representations.

The unification program I

The evidence provided by my results shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories.

I thus plan to continue researching along these lines to further develop this unification program. Central themes in this project will be:

- Deriving specific Morita-equivalences from the common mathematical practice
- Introducing new methods for generating Morita-equivalences
- Introducing new topos-theoretic invariants admitting natural characterizations
- Compiling a sort of 'encyclopedia of invariants and their characterizations' so that the 'working mathematician' can easily identify properties of theories and toposes which directly relate to his questions of interest
- Applying these methods in specific situations of interest in classical mathematics
- Automatizing the methodology 'toposes as bridges' on a computer to generate new and non-trivial mathematical results in a mechanical way

The unification program II

As explained above, the scope of application of the unification principles is extremely broad within Mathematics, but it also extends beyond it.

In fact, if successful, the unification project will likely have a significant impact on a variety of subjects including:

- Physics (e.g., analysis and interpretation of dualities, relativity theory and its relationship with quantum mechanics)
- Computer science (e.g., semantics of programming languages and automated theorem proving)
- Music Theory (e.g., analysis of composition, interpretation and performance)
- Linguistics (e.g., syntax and semantics of natural languages, comparative studies and the theory of translation)
- Philosophy (e.g., methodology of science, ontology of mathematical concepts)

For further reading

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- O. Caramello.
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  University of Cambridge, 2009
- My website: www.oliviacaramello.com