A topos-theoretic approach to Stone-type dualities

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Stone-type dualities

Consider the following 'Stone-type dualities':

- Stone duality for distributive lattices (and Boolean algebras)
- Lindenbaum-Tarski duality for atomic complete Boolean algebras
- The duality between spatial frames and sober spaces
- M. A. Moshier and P. Jipsen's topological duality for meet-semilattices
- Alexandrov equivalence between preorders and Alexandrov spaces
- Birkhoff duality for finite distributive lattices
- The duality between algebraic lattices and sup-semilattices
- The duality between completely distributive algebraic lattices and posets

They are all instances of just one topos-theoretic phenomenon.

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A machinery for generating dualities

- In this talk we present a general topos-theoretic machinery for generating dualites or equivalences between categories of preordered structures and categories of posets, locales or topological spaces.

- All of the above-mentioned dualities are recovered as the result of applying the machinery to particular sets of 'ingredients', and new dualities are established.

- In fact, infinitely many new dualities can be generated through the machinery in an essentially automatic way.

- The machinery is interesting because of its inherent technical flexibility; there are essentially four degrees of freedom in choosing the ingredients.

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Grothendieck topologies on preorders

Definition

Let $\mathscr{C}$ be a preorder.

(i) A (basis for a) Grothendieck topology on $\mathscr{C}$ is a function J which assigns to every element $c \in \mathscr{C}$ a family $J(c)$ of lower subsets of $(c) \downarrow$, called the *J-covers* on c, such that for any $S \in J(c)$ and any $c' \leq c$ the subset $S_{c'} = \{d \leq c' \mid d \in S\}$ belongs to $J(c')$.

(ii) A preorder site is a pair $(\mathscr{C}, J)$, where $\mathscr{C}$ is a preorder and J is a Grothendieck topology on $\mathscr{C}$.

(iii) A Grothendiek topology J on $\mathscr{C}$ is subcanonical if for every $c \in \mathscr{C}$ and any subset $S \in J(c)$, c is the supremum in $\mathscr{C}$ of the elements $d \in S$ (i.e., for any element c' in $\mathscr{C}$ such that for every $d \in S$ $d \leq c'$, we have $c \leq c'$).

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Examples of Grothendieck topologies

- If P is a preorder, the trivial topology on P is the one in which the only covers are the maximal ones.

- If D is a distributive lattice, the coherent topology on D is the one in which the covers are exactly those which contain finite families whose join is the given element.

- If F is a frame, the canonical topology on F is the one in which the covers are exactly the families whose join is the given element.

- If D is a disjunctively distributive lattice, the disjunctive topology on D is the one in which the covers are exactly those which contain finite families of pairwise disjoint elements whose join is the given element.

- If U is a k-frame, the k-covering topology on U is the one in which the covers are the those which contain families of less than k elements whose join is the given element.

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J-ideals

Definition

Given a preorder site $(\mathscr{C}, J)$, a J-ideal on $\mathscr{C}$ is a subset $I \subseteq \mathscr{C}$ such that

- for any $a, b \in \mathscr{C}$ such that $b \leq a$ in $\mathscr{C}$, $a \in I$ implies $b \in I$, and
- for any J-cover R on an element c of $\mathscr{C}$, if $a \in I$ for every $a \in R$ then $c \in I$.

We denote by $Id_J(\mathscr{C})$ the set of all the J-ideals on $\mathscr{C}$.

Theorem

Let $\mathscr{C}$ be a preorder and J be a Grothendieck topology on $\mathscr{C}$. Then $(Id_J(\mathscr{C}), \subseteq)$ is a frame.

Remark

If J is subcanonical (i.e. all the principal ideals on $\mathscr{C}$ are J-ideals) and $\mathscr{C}$ is a poset then we have an embedding $\mathscr{C} \hookrightarrow Id_J(\mathscr{C})$, which identifies $\mathscr{C}$ with the set of principal ideals on $\mathscr{C}$.

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The underlying philosophy

- In my paper

  The unification of Mathematics via Topos Theory

  I give a set of principles and methodologies which justify a view of Grothendieck toposes as 'bridges' for transferring information between distinct mathematical theories.
- This work represents a faithful implementation of this philosophy in a particular context.
- In fact, we establish our dualities precisely by 'functorializing' different representations of a given topos, which thus acts as a 'bridge' connecting the two sites:

$$(\mathscr{C}, J) \qquad\qquad\qquad Id_J(\mathscr{C})$$

$$\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(Id_J(\mathscr{C}))$$

Remark

For any preorder site $(\mathscr{C}, J)$, the J-ideals on $\mathscr{C}$ correspond precisely to the subterminal objects of the topos $\mathbf{Sh}(\mathscr{C}, J)$.

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Functorialization I

We can generate covariant or controvariant equivalences with categories of posets by appropriately functorializing the assignments above.

Definition

A morphism of sites $(\mathscr{C}, J) \to (\mathscr{D}, K)$, where $\mathscr{C}$ and $\mathscr{D}$ are meet-semilattices, is a meet-semilattice homomorphism $\mathscr{C} \to \mathscr{D}$ which sends J-covers to K-covers.

Theorem

1. *A morphism of sites $f : (\mathscr{C}, J) \to (\mathscr{D}, K)$ induces, naturally in f, a frame homomorphism $\dot{f} : Id_J(\mathscr{C}) \to Id_K(\mathscr{D})$.*
   This homomorphism sends a J-ideal I on $\mathscr{C}$ to the smallest K-ideal on $\mathscr{D}$ containing the image of I under f.

2. *If J and K are subcanonical then a frame homomorphism $Id_J(\mathscr{C}) \to Id_K(\mathscr{D})$ is of the form $\dot{f}$ for some f if and only if it sends principal ideals to principal ideals; if this is the case then f is isomorphic to the restriction of $\dot{f}$ to the principal ideals.*

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Functorialization II

Theorem

Let $\mathscr{C}$ and $\mathscr{D}$ be two preorders. Then

1. *For any monotone map $f : \mathscr{C} \to \mathscr{D}$, the map $B_f : \text{Id}(\mathscr{D}) \to \text{Id}(\mathscr{C})$ sending an ideal I on $\mathscr{D}$ to the inverse image $f^{-1}(I)$ of I under f is a frame homomorphism.*

2. *A frame homomorphism $F : \text{Id}(\mathscr{D}) \to \text{Id}(\mathscr{C})$ is of the form B_f for some monotone map $f : \mathscr{C} \to \mathscr{D}$ if and only if F preserves arbitrary infima, equivalently if and only if it has a left adjoint $F_! : \text{Id}(\mathscr{C}) \to \text{Id}(\mathscr{D})$, given by the formula $F_!(I) = \bigcap_{I \subseteq F(I')} I'$ (for any $I \in \text{Id}(\mathscr{C})$).*

3. *If $\mathscr{C}$ and $\mathscr{D}$ are posets then any monotone map $f : \mathscr{C} \to \mathscr{D}$ can be recovered from B_f as the restriction of its left adjoint $(B_f)_!$ to the subsets of principal ideals.*

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The general framework

We only discuss for simplicity the case of covariant equivalences with categories of frames, the other cases being conceptually similar to it.

Let $\mathscr{K}$ be a category of preordered structures, and suppose to have equipped each structure $\mathscr{C}$ in $\mathscr{K}$ with a Grothendieck topology $J_{\mathscr{C}}$ on $\mathscr{C}$ in such a way that every arrow $f : \mathscr{C} \to \mathscr{D}$ in $\mathscr{K}$ gives rise to a morphism of sites $f : (\mathscr{C}, J_{\mathscr{C}}) \to (\mathscr{D}, J_{\mathscr{D}})$.

These choices automatically induce a functor

$$A : \mathscr{K} \to \mathbf{Frm}$$

to the category **Frm** of frames sending any $\mathscr{C}$ in $\mathscr{K}$ to $Id_{J_{\mathscr{C}}}(\mathscr{C})$ and any $f : \mathscr{C} \to \mathscr{D}$ in $\mathscr{K}$ to the frame homomorphism $\dot{f} : Id_{J_{\mathscr{C}}}(\mathscr{C}) \to Id_{J_{\mathscr{D}}}(\mathscr{D})$.

Theorem

With the above notation, if all the Grothendieck topologies $J_{\mathscr{C}}$ are subcanonical and the preorders in $\mathscr{K}$ are posets then the functor $A : \mathscr{K} \to \mathbf{Frm}$ yields an isomorphism of categories between $\mathscr{K}$ and the subcategory of $\mathbf{Frm}$ given by the image of A.

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Recovering the structures through invariants

- The theorem just stated provides us with an infinite number of dualities. Still, it would be desirable to have a duality of $\mathscr{K}$ with a subcategory of **Frm** which is closed under isomorphisms in **Frm** (namely, the closure $ExtIm(A)$ of the image of A under isomorphisms in **Frm**) so that its objects (and arrows) could admit an intrinsic description in frame-theoretic terms.

- To achieve this, we investigate the problem of recovering a preorder $\mathscr{C}$ in $\mathscr{K}$ from the topos $\mathbf{Sh}(\mathscr{C}, J_{\mathscr{C}})$ (equivalently, from the frame $Id_{J_{\mathscr{C}}}(\mathscr{C})$) through an invariant, functorially in $\mathscr{C}$.

- It turns out that if the topologies $J_{\mathscr{C}}$ can be 'uniformly described through an invariant' C then the principal ideals on $\mathscr{C}$ can be characterized among the elements of the frame $Id_{J_{\mathscr{C}}}(\mathscr{C})$ precisely as the ones which are C-compact.

- This enables us to define a functor on the category $ExtIm(A)$ which yields, together with A, the desired equivalence.

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Topologies defined through invariants

Definition
Let *C* be a frame-theoretic invariant property of families of
elements of a frame (for example: to be finite, to be a singleton, to
be of cardinality at most *k* for some cardinal *k*, to be formed by
elements which are pairwise disjoint, to be directed etc.)

- Given a structure $\mathscr{C}$ in $\mathscr{K}$, the Grothendieck topology $J_\mathscr{C}$ is
  said to be *C-induced* if for any J_{can}^F-dense monotone
  embedding $i : \mathscr{C} \hookrightarrow F$ into a frame *F* (where J_{can}^F is the
  canonical topology on *F*), possibly satisfying some invariant
  property *P* which is known to hold for the canonical
  embedding $\mathscr{C} \hookrightarrow Id_{J_\mathscr{C}}(\mathscr{C})$, such that the $J_\mathscr{C}$-covers on $\mathscr{C}$ are
  sent by *i* to covers in *F*, for any family $\mathscr{A}$ of elements in $\mathscr{C}$
  there exists a $J_\mathscr{C}$-cover *S* on an element $c \in \mathscr{C}$ such that the
  elements $a \in \mathscr{A}$ such that $a \leq c$ generate *S* if and only if the
  image $i(\mathscr{A})$ of the family $\mathscr{A}$ in *F* has a refinement satisfying
  C made of elements of the form $i(c')$ (for $c' \in \mathscr{C}$).

- An element *u* of a frame *F* is said to be *C-compact* if every
  covering of *u* in *F* has a refinement satisfying *C*.

The main result

Theorem

If all the Grothendieck topologies $J_\mathscr{C}$ associated to the structures $\mathscr{C}$ in $\mathscr{K}$ are C-induced and the invariant C satisfies the property that for any structure $\mathscr{C}$ in $\mathscr{K}$ and for any family $\mathscr{F}$ of principal $J_\mathscr{C}$-ideals on $\mathscr{C}$, $\mathscr{F}$ has a refinement satisfying C (if and) only if it has a refinement satisfying C made of principal $J_\mathscr{C}$-ideals on $\mathscr{C}$ then the functor $ExtIm(A) \to \mathscr{K}$ sending a frame F in ExtIm(A) to the poset of C-compact elements of F and acting on the arrows accordingly is a categorical inverse to A.

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The target categories of frames

Theorem

- *The frames in ExtIm(A) are precisely the frames F with a basis B_F of C-compact elements which, regarded as a poset with the induced order, belongs to $\mathscr{K}$, and such that the embedding $B_F \hookrightarrow F$ satisfies property P, the property that every covering in F of an element of B_F is refined by a covering made of elements of B_F which satisfies the invariant C, and the property that the J_{B_F}-covering sieves are sent by the embedding $B_F \hookrightarrow F$ into covering families in F (where J_{B_F} is the Grothendieck topology with which B_F comes equipped as a structure in $\mathscr{K}$).*

- *The arrows $F \to F'$ in ExtIm(A) are the frame homomorphisms which send C-compact elements to C-compact elements in such a way that their restriction to the subsets of C-compact elements can be identified with an arrow in $\mathscr{K}$.*

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The subterminal topology

The following notion provides a way for endowing a given set of
points of a topos with a natural topology.

Definition
Let $\xi : X \to P$ be an indexing of a set P of points of a Grothendieck
topos $\mathscr{E}$ by a set X. We define the subterminal topology $\tau_\xi^{\mathscr{E}}$ as the
image of the function $\phi_{\mathscr{E}} : \mathrm{Sub}_{\mathscr{E}}(1) \to \mathscr{P}(X)$ given by

$$\phi_{\mathscr{E}}(u) = \{x \in X \mid \xi(x)^*(u) \cong 1_{\mathbf{Set}}\} \, .$$

We denote the topological space obtained by endowing the set X
with the topology $\tau_\xi^{\mathscr{E}}$ by $X_{\tau_\xi^{\mathscr{E}}}$.

The interest of this notion lies in its level of generality, as well as in
its formulation as a topos-theoretic invariant admitting a 'natural
behaviour' with respect to sites. Moreover, the following fact will be
crucial for us.

Fact
*If P is a separating set of points for $\mathscr{E}$ (for example, the set of all the
points of a localic topos having enough points) then the frame
$\mathscr{O}(X_{\tau_\xi^{\mathscr{E}}})$ of open sets of the space $X_{\tau_\xi^{\mathscr{E}}}$ is isomorphic (via $\phi_{\mathscr{E}}$) to the
frame $\mathrm{Sub}_{\mathscr{E}}(1)$ of subterminals of the topos $\mathscr{E}$.*

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Examples of subterminal topologies I

Definition

Let $(\mathscr{C}, \leq)$ be a preorder. A *J-prime filter* on $\mathscr{C}$ is a subset $F \subseteq \mathscr{C}$ such that F is non-empty, $a \in F$ implies $b \in F$ whenever $a \leq b$, for any $a, b \in F$ there exists $c \in F$ such that $c \leq a$ and $c \leq b$, and for any *J*-covering sieve $\{a_i \to a \mid i \in I\}$ in $\mathscr{C}$ if $a \in F$ then there exists $i \in I$ such that $a_i \in F$.

Theorem

Let $\mathscr{C}$ be a preorder and J be a Grothendieck topology on it. Then the space $X_{\tau\mathbf{Sh}(\mathscr{C},J)}$ has as set of points the collection $\mathscr{F}_{\mathscr{C}}^{J}$ of the J-prime filters on $\mathscr{C}$ and as open sets the sets the form

$$\mathscr{F}_I = \{F \in \mathscr{F}_{\mathscr{C}}^{J} \mid F \cap I \neq \emptyset\},$$

where I ranges among the J-ideals on $\mathscr{C}$. In particular, a sub-basis for this topology is given by the sets

$$\mathscr{F}_c = \{F \in \mathscr{F}_{\mathscr{C}}^{J} \mid c \in F\},$$

where c varies among the elements of $\mathscr{C}$.

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Examples of subterminal topologies II

- The Alexandrov topology ($\mathscr{E} = [\mathscr{P}, \textbf{Set}]$, where $\mathscr{P}$ is a preorder and ξ is the indexing of the set of points of $\mathscr{E}$ corresponding to the elements of $\mathscr{P}$)

- The Stone topology for distributive lattices ($\mathscr{E} = \textbf{Sh}(\mathscr{D}, J_{coh})$ and ξ is an indexing of the set of all the points of $\mathscr{E}$, where $\mathscr{D}$ is a distributive lattice and J_{coh} is the coherent topology on it)

- A topology for meet-semilattices ($\mathscr{E} = [\mathscr{M}^{\text{op}}, \textbf{Set}]$ and ξ is an indexing of the set of all the points of $\mathscr{E}$, where $\mathscr{M}$ is a meet-semilattice)

- The space of points of a locale ($\mathscr{E} = \textbf{Sh}(L)$ for a locale L and ξ is an indexing of the set of all the points of $\mathscr{E}$)

- A logical topology ($\mathscr{E} = \textbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the classiying topos of a geometric theory $\mathbb{T}$ and ξ is any indexing of the set of all the points of $\mathscr{E}$ i.e. models of $\mathbb{T}$)

- The Zariski topology

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Dualities with categories of topological spaces

- By using the subterminal topology, we can 'lift' the equivalences with frames established above to dualities with topological spaces, provided that the toposes involved have enough points.

- Indeed, the construction of the subterminal topology can be naturally made functorial.

- Thus, by assigning sets of points of the toposes corresponding to the structures in a natural way, we obtain a functor $\tilde{A} : \mathscr{K} \to \mathbf{Top}^{\mathrm{op}}$ such that $\mathscr{O} \circ \tilde{A} \cong A$, where $\mathscr{O} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Frm}$ the usual functor taking the frame of open sets of a topological space:

$$
\begin{array}{ccc}
& & \mathbf{Top}^{\mathrm{op}} \\
& \tilde{A} \nearrow & \downarrow \mathscr{O} \\
\mathscr{K} & \xrightarrow{\quad A \quad} & \mathbf{Frm}
\end{array}
$$

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The power of the machinery

- Functorializing general equivalences $\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{D}, K)$ (where $\mathscr{C}$ is a K-dense subcategory of $\mathscr{D}$ and J is induced by K on $\mathscr{C}$), we are able to recover all the dualities mentioned at the beginning of the talk as special cases generated through our machinery.

- At the same time, our framework allows enough flexibility to construct many new interesting dualities with particular properties.

- In fact, we essentially have four degrees of freedom:
    - (i) The choice of the structures $\mathscr{C}$;
    - (ii) The choice of the structures $\mathscr{D}$;
    - (iii) The choice of the topologies J and K;
    - (iv) The choice of points of the toposes $\mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{D}, K)$.

New dualities I

Among the new dualities that we obtain though our machinery, we
have:

- A duality between the category of meet-semilattices and
  meet-semilattices homomorphisms betweeen them and the
  category of locales whose objects are the locales with a basis of
  supercompact elements which is closed under finite meets and
  whose arrows are the locale maps whose associated frame
  homomorphisms send supercompact elements to supercompact
  elements.
- A duality between the category of frames with a basis of
  supercompact elements and complete homomorphisms between
  them and the category of posets (endowed with the Alexandrov
  topology).
  This duality restricts to Lindenbaum-Tarski duality.
- A duality between the category of disjunctively distributive
  lattices and the category whose objects are the sober topological
  spaces which have a basis of disjunctively compact open sets
  which is closed under finite intersection and satisfies the
  property that any covering of a basic open set has a disjunctively
  compact refinement by basic open sets and whose arrows are
  the continuous maps between such spaces such that the inverse
  image of any disjunctively compact open set is a disjunctively
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New dualities II

- For any regular cardinal k, a duality between the category of k-frames and the category whose objects are the frames which have a basis of k-compact elements which is closed under finite meets and whose arrows are the frame homomorphisms between them which send k-compact elements to k-compact elements.

- A duality between the category of disjunctive frames and the category $\mathbf{Pos}_{dis}$ which has as objects the posets $\mathscr{P}$ such that for any $a, b \in \mathscr{P}$ there exists a family $\{c_i \mid i \in I\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $p \leq a$ and $p \leq b$ if and only if $p \leq c_i$ for a unique $i \in I$ and as arrows $\mathscr{P} \to \mathscr{P}'$ the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ such that for any $b \in \mathscr{P}'$ there exists a family $\{c_i \mid i \in I\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $g(p) \leq b$ if and only if $p \leq c_i$ for a unique $i \in I$.

New dualities III

- A duality between the category **DirIrrPFrm** of directly generated preframes whose objects are the directly generated preframes and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the preframe homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category **Pos**$_{dir}$ having as objects the posets $\mathscr{P}$ such that for any $a, b \in \mathscr{P}$ there is $c \in \mathscr{P}$ such that $c \leq a$ and $c \leq b$ and for any elements $d, e \in \mathscr{P}$ such that $d, e \leq a$ and $d, e \leq b$ there exists $z \in \mathscr{P}$ such that $z \leq a$, $z \leq b$, $d, e \leq z$, and as arrows $\mathscr{P} \to \mathscr{P}'$ the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ with the property that for any $b \in \mathscr{P}'$ there exists $a \in \mathscr{P}$ such that $g(a) \leq p$ and for any two $u, v \in \mathscr{P}$ such that $g(u) \leq b$ and $g(v) \leq b$ there exists $z \in \mathscr{P}$ such that $u, v \leq z$ and $g(z) \leq b$.
  This duality restricts to the duality between algebraic lattices and sup-semilattices.

- An equivalence between the category of meet-semilattices and the category whose objects are the the meet-semilattices F with a bottom element 0_F which have the property that for any $a, b \in F$ with $a, b \neq 0$, $a \wedge b \neq 0$ and whose arrows are the meet-semilattice homomorphisms $F \to F'$ which send 0_F to $0_{F'}$ and any non-zero element of F to a non-zero element of F'.

New dualities IV

- A duality between the category **IrrDLat** whose objects are the irreducibly generated distributive lattices and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the distributive lattices homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category **Pos**$_{comp}$ whose objects are the posets and whose arrows $\mathscr{P} \to \mathscr{P}'$ are the monotone maps $g : \mathscr{P} \to \mathscr{P}'$ such that for any $q \in \mathscr{P}'$, there exists a finite family $\{a_k \mid k \in K\}$ of elements of $\mathscr{P}$ such that for any $p \in \mathscr{P}$, $g(p) \leq q$ if and only if $p \leq a_k$ for some $k \in K$.
  This duality restricts to Birkhoff duality.

- A duality between the category **AtDLat** whose objects are the atomic distributive lattices and whose arrows $\mathscr{D} \to \mathscr{D}'$ are the distributive lattices homomorphisms $f : \mathscr{D} \to \mathscr{D}'$ between them such that the frame homomorphism $A(f) : Id_{J_{\mathscr{D}}}(\mathscr{D}) \to Id_{J'_{\mathscr{D}}}(\mathscr{D}')$ which sends an ideal I of $\mathscr{D}$ to the ideal of $\mathscr{D}'$ generated by $f(I)$ preserves arbitrary infima, and the category **Set**$_f$ whose objects are the sets and whose arrows $A \to B$ are the functions $f : A \to B$ such that the inverse image under f of any finite subset of B is a finite subset of A.

- ...

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Other applications

A great amount of applications can be established, besides the construction of new dualities, by applying the technique 'toposes as bridges' to the equivalences of toposes considered above. Examples include:

- Adjunctions between categories of preorders and categories of frames or locales; for example, between meet-semilattices (resp. distributive lattices, preframes, Boolean algebras) and frames

- Translations of properties of preordered structures into properties of the corresponding locales or topological spaces (for example, characterizations of the Stone-type spaces associated to the structures which are trivial, almost discrete, extremally disconnected etc.)

- Representation theorems for preordered structures

- Priestley-type dualities for various kinds of preordered structures

- Completeness theorems for propositional logics

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Future directions

The theory just described paves the way for a vast world of new possibilities. A few natural directions that one could immediately pursue are the following:

- Investigation of particular dualities generated through the machinery; this should be interesting because, as it is clear from our interpretation, the 'generic' duality generated through our method has essentially the same level of 'mathematical depth' as the classical Stone duality.

- 'Automatic' generation of new dualities by using our machinery, in the preordered context and possibly beyond it.

- Translations of properties across the dualities, by applying the methodology 'toposes as bridges' to appropriate topos-theoretic invariants; again, this translation can be performed automatically in many cases, but the results generated in this way will in general be non-trivial.

- Use of this framework for generating dualities for more complex algebraic or topological structures (e.g. Priestley-type dualities) through the identification of appropriate topos-theoretic invariants.

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