The idea of bridge and its unifying role in science

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Unifying Mathematics

- Set Theory has represented the first serious attempt of Logic to unify Mathematics at least at the level of language.
- Later, Category Theory offered an alternative abstract language in which most of Mathematics can be formulated.

Anyway, both these systems realize a unification which is still limited in scope, in the sense that, even though each of them provides a way of expressing and organizing Mathematics in one single language, they do not offer by themselves effective methods for an actual transfer of knowledge between distinct fields.

Instead, the methodologies introduced in the paper "The unification of Mathematics via Topos Theory" define a different and more substantial approach to the problem of 'unifying Mathematics'.

The eclectic nature of toposes

In this talk, whenever I use the word 'topos', I really mean 'Grothendieck topos'.

A Grothendieck topos can be seen as:

- a generalized space
- a mathematical universe
- a theory modulo 'Morita-equivalence'

In this talk, I will describe the philosophical principles underlying a new view of toposes as unifying spaces which can serve as bridges for transferring information, ideas and results between distinct mathematical theories.

This approach, first introduced in my Ph.D. thesis, has already generated a great number of non-trivial applications into distinct mathematical fields and points towards a realization of Topos Theory as a unifying theory of Mathematics.

Toposes as *bridges* I

- To any mathematical theory $\mathbb{T}$ (of a general specified form) one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called the classifying topos of the theory, which represents its 'semantical core'.
- Conversely, every Grothendieck topos arises as the classifying topos of some theory.
- Two mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is if and only if they are indistinguishable from a semantic point of view; such theories are said to be Morita-equivalent.
- So a topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.
- The notion of Morita-equivalence formalizes in many situations the feeling of 'looking at the same thing in different ways', and in fact it is ubiquitous in Mathematics.

Toposes as *bridges* II

- The existence of different theories with the same classifying topos translates, at the technical level, into the existence of different representations for the same topos.
- Topos-theoretic invariants can thus be used to transfer information from one theory to another:

$$\begin{tikzcd}
 & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed] & \\
\mathbb{T} \arrow[ur, dashed] & & \mathbb{T}'
\end{tikzcd}$$

- The transfer of information takes place by expressing a given invariant in terms of the different representation of the topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

Toposes as *bridges* III

- These methodologies are technically effective because the relationship between a topos and its representations is often very natural, enabling us to easily transfer invariants across different representations (and hence, between different theories).
- As a matter of fact, these methods define a new way of doing Mathematics which is 'upside-down' compared with the 'usual' one. Indeed, one assumes as primitive ingredients abstract notions such as Morita-equivalences and topos-theoretic invariants, and proceeds to extract from them concrete information relevant for classical mathematics.
- Moreover, there is an strong element of automatism in these techniques; by means of them, one can generate new mathematical results without really making any creative effort. Indeed, in many cases one can just readily apply the general characterizations expressing topos-theoretic invariants on a topos in terms of properties of its representations to the particular Morita-equivalence under consideration.

Why toposes?

- Generality: Unlike most of the invariants used in Mathematics, the level of generality of topos-theoretic invariants is such to make them suitable for comparing with each other (first-order) mathematical theories of essentially any kind.
- Expressivity: On the other hand, many important invariants arising in Mathematics can be expressed as topos-theoretic invariants.
- Centrality: The fact that topos-theoretic invariants often specialize to important properties or constructions of natural mathematical or logical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole.
- Technical flexibility: Toposes are mathematical universes which are very rich in terms of internal structure; moreover, they have a very-well behaved representation theory, which makes them extremely effective computationally when considered as 'bridges'.

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Unifying Mathematics

Toposes as bridges

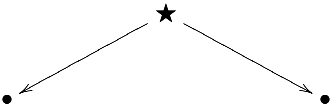
The idea of bridge

For further reading

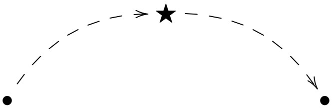
The concept of unification

We can distinguish between two different kinds of unification.

- ‘Static’ unification (through generalization): two concepts are seen to be special instances of a more general one:



- ‘Dynamic’ unification (through construction): two objects are related to each other through a third one (usually constructed from each of them), which acts like a ‘bridge’ enabling a transfer of information between them.



The transfer of information arises from the process of ‘translating’ properties (resp. constructions) on the ‘bridge object’ into properties (resp. constructions) on the two objects.

Comparing different objects

- One is generally interested in comparing pairs of objects between which there is some kind of relationship. Let us suppose, as it normally happens, that this relationship can be formalized as an equivalence relation on the set of objects.
- In order to transfer information between objects related by an equivalence relation (that is, belonging to the same equivalence class), it is thus of fundamental importance to identify (and, possibly, classify) the properties of the objects that are invariant with respect to the equivalence relation.
- Depending on the cases, this can be an approachable task or an hopelessly difficult one.

Realizing equivalence classes I

Definition

Given two sets I and O, and two equivalence relations $\simeq_I$ and $\simeq_O$ respectively on I and on O, an invariant construction $f : (I, \simeq_I) \to (O, \simeq_O)$ is a function $f : I \to O$ which respects the equivalence relations (i.e. such that whenever $x \simeq_I y$, we have $f(x) \simeq_O f(y)$).

We say that f is conservative if it reflects the equivalence relations (i.e. whenever $f(x) \simeq_O f(y)$, we have $x \simeq_I y$).

Definition

Given an invariant construction $f : (I, \simeq_I) \to (O, \simeq_O)$, a bridge connecting two objects $x, y \in I$ such that $x \simeq_I y$ is an object $b \in O$ such that $b \simeq_O f(x)$ and $b \simeq_O f(y)$:

$$\begin{tikzcd} & f(x) \simeq_O b \simeq_O f(y) & \\ x \arrow[ur, dash, dashed] & & y \arrow[ul, dash, dashed] \end{tikzcd}$$

Realizing equivalence classes II

- Given a conservative invariant construction $f : (I, \simeq_I) \to (O, \simeq_O)$, bridge objects in O, considered up to $\simeq_O$-equivalence, can be thought of as classifying objects, since they can be seen as canonical representatives of $\simeq_I$-equivalence classes.
- The importance of bridges lies in the fact that, provided that the invariant construction $f : I \to O$ is well-behaved enough to allow appropriate 'unravellings' of properties (resp. constructions) on $f(z)$ in terms of properties (resp. constructions) on z (for objects z of type I), they can be used as means for transferring information between objects of type I which are in relation $\simeq_I$ with each other.
- Of course, having a bridge is most useful in classifying $\simeq_I$-invariant properties in cases in which it is more manageable to work with objects of type O than with objects of type I, or when the relation $\simeq_O$ is more tractable than the relation $\simeq_I$.

Structural translations

The method of bridges can be interpreted linguistically as a methodology for translating concepts from one context to another.

But which kind of translation is this?

In general, we can distinguish between two essentially different approaches to translation.

- The 'dictionary-oriented' or 'bottom-up' approach, consisting in a dictionary-based renaming of the single words composing the sentences.
- The 'invariant-oriented' or 'top-down' approach, consisting in the identification of appropriate concepts that should remain invariant in the translation, and in the subsequent analysis of how these invariants can be expressed in the two languages.

The 'bridge-based' translations, and in particular the topos-theoretic ones, are of the latter kind.

In the topos-theoretic case, the invariant properties are topos-theoretic invariants defined on toposes, and the expression of these invariants in terms of the two different theories is essentially determined by the structural relationship between the topos and its two different representations.

Ideal = real?

- Bridges abound both in Mathematics and in other scientific fields, and can be considered ‘responsible’ (at least abstractly) for the genesis of things and the nature of reality as we perceive it.
- The idea of bridge is an abstraction, but, interestingly, bridges arising in the experimental sciences can often be identified with actual physical objects.
- In fact, the most enlightening situations occur when these ideal objects admit ‘concrete’ representations, allowing us to contemplate the dynamics of ‘differentiation from the unity’ in all its aspects.
- Topos Theory allows us to materialize a tremendous number of ideal objects, and hence to establish effective bridges between a great variety of different contexts.
- In general, looking for ‘concrete’ representations of (or ways of realizing) imaginary concepts can lead to the discovery of more ‘symmetric’ environments in which phenomena can be described in natural and unified ways.

For further reading

O. Caramello.
The unification of Mathematics via Topos Theory,
arXiv:math.CT/1006.3930, 2010

O. Caramello.
The Duality between Grothendieck toposes and Geometric Theories, Ph.D. thesis
University of Cambridge, 2009

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