Fraïssé's construction from a topos-theoretic perspective

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Grothendieck toposes

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space X by a pair $(\mathscr{C}, J)$ consisting of a (small) category $\mathscr{C}$ and a 'generalized notion of covering' J on it, and taking sheaves (in a generalized sense) over the pair:

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{Sh}(X) \\ \rightsquigarrow & & \rightsquigarrow \\ (\mathscr{C}, J) & \dashrightarrow & \mathbf{Sh}(\mathscr{C}, J) \end{array}$$

Geometric theories

Definition

- A geometric formula over a signature Σ is any formula (with a finite number of free variables) built from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.
- A geometric theory over a signature Σ is any theory whose axioms are of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are geometric formulae over Σ and $\vec{x}$ is a context suitable for both of them.

Fact

Most of the theories naturally arising in Mathematics are geometric; and if a finitary first-order theory is not geometric, we can always associate to it a finitary geometric theory over a larger signature (the so-called Morleyization of the theory) with essentially the same models in the category **Set** *of sets.*

The notion of classifying topos

Definition

Let $\mathbb{T}$ be a geometric theory over a given signature. A classifying topos of $\mathbb{T}$ is a Grothendieck topos $\mathbf{Set}[\mathbb{T}]$ such that for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\mathbf{Geom}(\mathscr{E}, \mathbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

natural in $\mathscr{E}$.

Theorem

Every geometric theory (over a given signature) has a classifying topos. Conversely, every Grothendieck topos arises as the classifying topos of some geometric theory.

The classifying topos of a geometric theory $\mathbb{T}$ can always be constructed canonically from the theory by means of a syntactic construction, namely as the topos of sheaves $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ on the geometric syntactic category $\mathscr{C}_{\mathbb{T}}$ of $\mathbb{T}$ with respect to the syntactic topology $J_{\mathbb{T}}$ on it (i.e. the canonical Grothendieck topology on $\mathscr{C}_{\mathbb{T}}$).

Toposes as bridges I

- In my paper

  The unification of Mathematics via Topos Theory

  I give a set of principles and methodologies which justify a view of Grothendieck toposes as 'bridges' for transferring information between distinct mathematical theories.
- The fundamental intuition underlying this view is that a given mathematical property can manifest itself in several different forms in the context of mathematical theories which have a common 'semantical core' but a different linguistic presentation.
- The remarkable fact is that if the property is formulated as a topos-theoretic invariant on some topos then the expression of it in terms of the different theories classified by the topos is determined to a great extent by the technical relationship between the topos and the different sites of definition for it.

Toposes as bridges II

- Indeed, the fact that different mathematical theories have equivalent classifying toposes translates into the existence of different sites of definition for one topos.
- Topos-theoretic invariants can then be used to transfer properties from one theory to another.
- This idea is technically feasible because the relationship between a site $(\mathscr{C}, J)$ and the topos $\mathbf{Sh}(\mathscr{C}, J)$ which it 'generates' is often very natural, enabling us to easily transfer invariants across different sites.
- A topos thus acts as a 'bridge' which allows the transfer of information and results between theories classified by the same topos:

$$
\begin{array}{ccc}
(\mathscr{C}, J) & & (\mathscr{C}', J') \\
& \searrow \quad \nearrow & \\
& \mathbf{Sh}(\mathscr{C}, J) \simeq \mathbf{Sh}(\mathscr{C}', J') &
\end{array}
$$

Subtoposes

Definition

A subtopos of a topos $\mathscr{E}$ is a geometric inclusion of the form $\mathbf{sh}_j(\mathscr{E}) \hookrightarrow \mathscr{E}$ for a local operator j on $\mathscr{E}$.

Fact

- *A subtopos of a topos $\mathscr{E}$ can be thought of as an equivalence class of geometric inclusions with codomain $\mathscr{E}$; hence, the notion of subtopos is a topos-theoretic invariant.*
- *If $\mathscr{E}$ is the topos $\mathbf{Sh}(\mathscr{C}, J)$ of sheaves on a site $(\mathscr{C}, J)$, the subtoposes of $\mathscr{E}$ are in bijective correspondence with the Grothendieck topologies J' on $\mathscr{C}$ which contain J (i.e. such that every J-covering sieve is J'-covering).*

The duality theorem

Definition

- Let $\mathbb{T}$ be a geometric theory over a signature Σ. A quotient of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ over Σ such that every axiom of $\mathbb{T}$ is provable in $\mathbb{T}'$.
- Let $\mathbb{T}$ and $\mathbb{T}'$ be geometric theories over a signature Σ. We say that $\mathbb{T}$ and $\mathbb{T}'$ are syntactically equivalent, and we write $\mathbb{T} \equiv_s \mathbb{T}'$, if for every geometric sequent σ over Σ, σ is provable in $\mathbb{T}$ if and only if σ is provable in $\mathbb{T}'$.

Theorem

Let $\mathbb{T}$ be a geometric theory over a signature Σ. Then the assignment sending a quotient of $\mathbb{T}$ to its classifying topos defines a bijection between the $\equiv_s$-equivalence classes of quotients of $\mathbb{T}$ and the subtoposes of the classifying topos **Set**$[\mathbb{T}]$ *of $\mathbb{T}$.*

Amalgamation and joint embedding properties

Definition

- A category $\mathscr{C}$ is said to satisfy the amalgamation property (AP) if for every objects $a, b, c \in \mathscr{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathscr{C}$ there exists an object $d \in \mathscr{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathscr{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{array}{ccc} a & \xrightarrow{\;f\;} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ c & \xrightarrow[\;g'\;]{} & d \end{array}$$

- A category $\mathscr{C}$ is said to satisfy the joint embedding property (JEP) if for every pair of objects $a, b \in \mathscr{C}$ there exists an object $c \in \mathscr{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathscr{C}$:

$$\begin{array}{ccc} & & a \\ & & \downarrow{\scriptstyle f} \\ b & \xrightarrow[\;g\;]{} & c \end{array}$$

Theories of presheaf type

Definition

- A geometric theory $\mathbb{T}$ over a signature Σ is said to be of presheaf type if it is classified by a presheaf topos.
- A model M of a theory of presheaf type $\mathbb{T}$ in the category **Set** is said to be finitely presentable if the functor $Hom_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M, -) : \mathbb{T}\text{-mod}(\mathbf{Set}) \to \mathbf{Set}$ preserves filtered colimits.

The class of theories of presheaf type contains all the cartesian (in particular, finitary algebraic) theories and many other significant mathematical theories.

Fact

For any theory of presheaf type $\mathscr{C}$, we have two different representations of its classifying topos:

$$[f.p.\mathbb{T}\text{-}mod(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

where $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ is the category of finitely presentable $\mathbb{T}$-models.

Note that this gives us a perfect opportunity to test the effectiveness of the philosophy 'toposes as bridges'!

The topos-theoretic interpretation I

Let $\mathbb{T}$ be a theory of presheaf type. If the category $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ satisfies AP then the subtopos

$$i_{at} : \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \hookrightarrow [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}]$$

(where J_{at} is the atomic topology on $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}$) transfers, via the equivalence above, to a subtopos of $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$, which can in turn be identified, via the duality theorem, with the canonical inclusion

$$i : \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \hookrightarrow \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

of the classifying topos of a unique quotient $\mathbb{T}'$ of $\mathbb{T}$ into the classifying topos of $\mathbb{T}$.

The topos-theoretic interpretation II

We thus obtain a commutative diagram

$$\begin{array}{ccc} [\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] & \xrightarrow{\simeq} & \mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}}) \\ \uparrow {\scriptstyle i_{at}} & & \uparrow {\scriptstyle i} \\ \mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) & \xrightarrow{\simeq} & \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \end{array} .$$

According to the philosophy 'toposes as bridges', we shall take the equivalence

$$\mathbf{Sh}(\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at}) \simeq \mathbf{Sh}(\mathscr{C}_{\mathbb{T}'}, J_{\mathbb{T}'})$$

as the starting point of our investigation, and proceed to extract information about it by considering various topos-theoretic invariants from the points of view of the two sites of definition of the given classifying topos.
The different properties involved in Fraïssé's theorem will be interpreted as different manifestations of a unique property lying at the topos-theoretic level.

Topos-theoretic Fraïssé's theorem

Theorem

*Let $\mathbb{T}$ be a theory of presheaf type such that the category f.p.$\mathbb{T}$-mod(**Set**) satisfies both amalgamation and joint embedding properties. Then any two countable homogeneous $\mathbb{T}$-models in* **Set** *are isomorphic.*

The quotient of $\mathbb{T}$ axiomatizing the homogeneous $\mathbb{T}$-models is precisely the Booleanization of $\mathbb{T}$, as defined below.

The result arises from the unification of different points of view on the same structures, obtained by using a certain classifying topos as a bridge connecting its different representations.

Atomicity

Definition

A Grothendieck topos is said to be atomic if all its subobject lattices are atomic complete Boolean algebras.

Theorem

Let $(\mathscr{C}, J)$ be a site. The topos $\mathbf{Sh}(\mathscr{C}, J)$ is atomic if and only if for every $c \in \mathscr{C}$ there exists a J-covering sieve on c generated by arrows f with the property that $\emptyset \notin J(dom(f))$ and for every arrow g which factors through f, either $\{k : dom(k) \to dom(f) \mid f \circ k \textit{ factors through } g\} \in J(dom(f))$ or $\emptyset \in J(dom(g))$. In particular:

- *If $\mathscr{C}^{op}$ satisfies AP and J is the atomic topology on $\mathscr{C}$ then $\mathbf{Sh}(\mathscr{C}, J)$ is atomic;*
- *If $(\mathscr{C}, J)$ is the syntactic site of a geometric theory $\mathbb{T}$ then $\mathbf{Sh}(\mathscr{C}, J)$ is atomic if and only if $\mathbb{T}$ is atomic, i.e. for every context $\vec{x}$ over the signature of $\mathbb{T}$, there is a set $B_{\vec{x}}$ of $\mathbb{T}$-complete geometric formulae in that context such that $\top \vdash_x \bigvee_{\phi \in B_{\vec{x}}} \phi$ is provable in $\mathbb{T}$ (where by $\mathbb{T}$-complete formula we mean a geometric formula $\phi(\vec{x})$ such that the sequent $\bot \vdash_{\vec{x}} \phi$ is not provable in $\mathbb{T}$, but for every geometric formula ψ in the same context either $\psi \vdash_{\vec{x}} \bot$ or $\phi \vdash_{\vec{x}} \psi$ is provable in $\mathbb{T}$).*

Two-valuedness

Definition

A Grothendieck topos is said to be two-valued if the only subobjects of the terminal object are the zero one and the identity one, and they are distinct from each other.

Theorem

Let $(\mathscr{C}, J)$ be a site. Then the topos $\mathbf{Sh}(\mathscr{C}, J)$ *is two-valued if and only if the only J-ideals on $\mathscr{C}$ are the trivial ones, and they are distinct from each other. In particular:*

- *If $\mathscr{C}^{op}$ is non-empty and satisfies AP and J is the atomic topology on $\mathscr{C}$ then* $\mathbf{Sh}(\mathscr{C}, J)$ *is two-valued if and only if $\mathscr{C}^{op}$ satisfies JEP.*
- *If $(\mathscr{C}, J)$ is the syntactic site of a geometric theory $\mathbb{T}$ then* $\mathbf{Sh}(\mathscr{C}, J)$ *is two-valued if and only if $\mathbb{T}$ is complete, i.e. for any geometric sentence ϕ over the signature of $\mathbb{T}$, either ϕ is $\mathbb{T}$-provably equivalent to $\bot$ or to $\top$, but not both.*

Theorem

Let $\mathbb{T}$ be a geometric theory. If $\mathbb{T}$ is complete and atomic then $\mathbb{T}$ is countably categorical, i.e. any two countable models of $\mathbb{T}$ in **Set** *are isomorphic.*

The theory of homogeneous models I

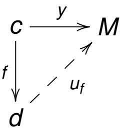
Definition

A point of a Grothendieck topos $\mathscr{E}$ is a geometric morphism $\mathbf{Set} \to \mathscr{E}$.

Theorem

Let $(\mathscr{C}, J)$ be a site. Then the points of the topos $\mathbf{Sh}(\mathscr{C}, J)$ correspond to the J-continuous flat functors on $\mathscr{C}$. In particular:

- *If $(\mathscr{C}, J)$ is the syntactic site of a geometric theory $\mathbb{T}$ then the points of the topos $\mathbf{Sh}(\mathscr{C}, J)$ correspond to the models of $\mathbb{T}$ in $\mathbf{Set}$.*
- *If $\mathscr{C}$ is the opposite of the category f.p.$\mathbb{T}$-mod($\mathbf{Set}$) (for a theory of presheaf type $\mathbb{T}$) and J is the atomic topology on $\mathscr{C}$ then the points of the topos $\mathbf{Sh}(\mathscr{C}, J)$ correspond to the homogeneous $\mathbb{T}$-models in $\mathbf{Set}$, i.e. to the models $M \in \mathbb{T}$-mod($\mathbf{Set}$) such that for each arrow $f : c \to d$ in $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ and arrow $y : c \to M$ in $\mathbb{T}\text{-}mod(\mathbf{Set})$ there exists an arrow $u_f : d \to M$ in $\mathbb{T}\text{-}mod(\mathbf{Set})$ such that $y = u_f \circ f$:*



The theory of homogeneous models II

Definition

The Booleanization of a Grothendieck topos $\mathscr{E}$ is the subtopos $\mathbf{sh}_{\neg\neg}(\mathscr{E})$ of $\mathscr{E}$.

Theorem

- *Let $\mathscr{C}$ be a category satisfying the amalgamation property. Then the Booleanization of the topos $[\mathscr{C}, \mathbf{Set}]$ can be identified with the topos $\mathbf{Sh}(\mathscr{C}^{op}, J)$ of sheaves on $\mathscr{C}^{op}$ with respect to the atomic topology on $\mathscr{C}^{op}$.*
- *Let $\mathbb{T}$ be a geometric theory. Then the Booleanization of its classifying topos $\mathbf{Sh}(\mathscr{C}_{\mathbb{T}}, J_{\mathbb{T}})$ can be identified, via the duality theorem, with the classifying topos of the Booleanization of $\mathbb{T}$, that is the quotient of $\mathbb{T}$ obtained from $\mathbb{T}$ by adding the axiom*

$$\top \vdash_{\vec{y}} \psi$$

for any stably consistent formula $\psi(\vec{y})$ with respect to $\mathbb{T}$ (i.e. a formula-in-context $\psi(\vec{y})$ such that for any geometric formula $\chi(\vec{y})$ in the same context such that $\chi \vdash_{\vec{y}} \bot$ is not provable in $\mathbb{T}$, $\chi \wedge \psi \vdash_{\vec{y}} \bot$ is not provable in $\mathbb{T}$).

The theory of homogeneous models III

Considering the invariant property of an object of a topos to be irreducible in connection with this equivalence, we can show that the category $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ is equivalent to the opposite of the full subcategory $\mathscr{C}_{\mathbb{T}}^{\text{irr}}$ of $\mathscr{C}_{\mathbb{T}}$ on the $\mathbb{T}$-irreducible formulae (i.e. the formulae which do not admit any non-trivial $J_{\mathbb{T}}$-covering sieves on them), via the assignment sending a $\mathbb{T}$-irreducible formula to the $\mathbb{T}$-model that it presents.

By using this equivalence

$$\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set}) \simeq {\mathscr{C}_{\mathbb{T}}^{\text{irr}}}^{\text{op}},$$

we get an alternative axiomatization of the Booleanization of $\mathbb{T}$, obtained by adding to $\mathbb{T}$ all the axioms of the form $\phi \vdash_{\vec{x}} (\exists \vec{y})\theta$, where $[\theta] : \{\vec{y} \,.\, \psi\} \to \{\vec{x} \,.\, \phi\}$ is any arrow in $\mathscr{C}_{\mathbb{T}}$ and $\phi(\vec{x})$, $\psi(\vec{y})$ are geometric formulae over the signature of $\mathbb{T}$ presenting some $\mathbb{T}$-model.

The unification

By integrating the different points of view using the classifying topos of $\mathbb{T}'$ as a bridge between its two different representation, we obtain the following insights, from which our main theorem follows at once.

- The fact that the category f.p.$\mathbb{T}$-mod(**Set**) satisfies AP implies that the theory $\mathbb{T}'$ is atomic.
- If the category f.p.$\mathbb{T}$-mod(**Set**) is non-empty then the theory $\mathbb{T}'$ is complete if and only if f.p.$\mathbb{T}$-mod(**Set**) satisfies JEP.
- The theory $\mathbb{T}'$ axiomatizes the homogeneous $\mathbb{T}$-models.
- The theory $\mathbb{T}'$ coincides with the Booleanization of $\mathbb{T}$.

Remark

If the theory $\mathbb{T}$ is coherent then the theory $\mathbb{T}'$ is coherent as well.

Further insights I

- Let $\phi(\vec{x})$ be a formula which presents a $\mathbb{T}$-model in **Set**; then $\phi(\vec{x})$ is $\mathbb{T}'$-complete, i.e. for every geometric formula ψ in the same context in the signature of $\mathbb{T}$ either $\psi \vdash_{\vec{x}} \bot$ or $\phi \vdash_{\vec{x}} \psi$ is provable in $\mathbb{T}'$, but not both).
- Let $\mathbb{T}$ be a theory of presheaf type and let $\mathbb{T}'$ be the theory of homogeneous $\mathbb{T}$-models. Suppose that $\mathbb{T}'$ has enough models and that all the arrows in $\mathbb{T}\text{-mod}(\mathbf{Set})$ are monic. Then the following conditions are equivalent.
    - The category $\text{f.p.}\mathbb{T}\text{-mod}(\mathbf{Set})$ satisfies AP;
    - For every finitely presentable $\mathbb{T}$ model c in **Set** there exists a $\mathbb{T}'$-model in **Set** (that is, a homogeneous $\mathbb{T}$-model in **Set**) and a $\mathbb{T}$-model homomorphism $c \to M$.

Further insights II

Theorem

Let $\mathbb{T}$ be a consistent coherent theory of presheaf type such that the category $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ satisfies AP. Then $\mathbb{T}$ has a consistent coherent quotient which is atomic and countably categorical.

Theorem

Let $\mathbb{T}$ be a theory of presheaf type such that $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ satisfies AP. If the topos $\mathbf{Sh}(f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}, J_{at})$ has enough points (for example when the theory $\mathbb{T}$ is coherent or the category $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})$ has all fc finite colimits) and there exists at least one $\mathbb{T}$-model in $\mathbf{Set}$, then there exists at least one homogeneous $\mathbb{T}$-model in $\mathbf{Set}$.

For further reading

- O. Caramello.
  Fraïssé's construction from a topos-theoretic perspective,
  arXiv:math.CT/0805.2778
- O. Caramello.
  The unification of Mathematics via Topos Theory,
  arXiv:math.CT/1006.3930
- My website:
  www.oliviacaramello.com